On the Periodic Evans Function and its Applications to the Nonlinear Schrödinger Equation

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Bachelor's thesis
2025:K24

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Lund, August 2025

Acknowledgements

I wish to express my deepest gratitude to my thesis advisor, Erik Wahlén, for introducing me to this fascinating topic and for his unwavering patience and support throughout my research journey. His guidance has been instrumental in navigating the challenges of this work, and I am sincerely grateful for his encouragement and willingness to address my numerous questions and errors.I am also thankful to Anders Olofsson, whose encouragement to pursue mechanics and differential equations provided clarity and direction at a time when my academic choices felt clouded by uncertainty.

Popular Science Description

Mathematical models describing natural phenomena have often given rise to seemingly cyclic, and repetitive patterns, which have manifested themselves in a wide-ranging set of topics. From population dynamics in a biological setting to the evolution of stellar structures and celestial dynamics, we have oftentimes witnessed wave-like behavior, or at least some permanent structure that follows a certain pattern. An immediate question that plays a pivotal role in applied mathematics is that of the *stability* of such behavior. For instance, when it comes to traveling water waves, the term stability refers to the wave maintaining its shape as it moves forward in time, while small disturbances diminish without causing major effects. However, non-linearity among a large number of interesting evolving systems makes it a challenging task to provide an answer right away. It is also what keeps mathematicians in business since the study of stability in a diverse set of applications demands the development of newer theories and feasible techniques. The central topic of this thesis revolves around one such theory, namely the *Evans function.*

One can conveniently think of this function as a stability detector device. Mathematically speaking, it connects two theoretical perspectives from functional analysis and dynamical systems on stability. It is named after the American mathematician John W. Evans, who originally trained as a medical doctor, became fascinated by mathematics, and eventually left his medical career to pursue a PhD in math. Having introduced the Evans function in the 1970s, his interest in mathematics was influenced by his medical background, particularly the Hodgkin-Huxley model for nerve impulse propagation, formulated about 20 years earlier. While Hodgkin and Huxley had shown that nerve impulses could travel as waves, Evans focused on proving their stability. Though he didn't fully solve this problem, he developed a theory with applications extending beyond neuroscience.

In this bachelor's thesis, the aim is to introduce the main ideas surrounding this method, and eventually see one of its many applications in stability analysis of solitons. In particular, we will be focusing on the Nonlinear Schrödinger (NLS) equation which, among other areas, appears frequently in quantum mechanics.

Abstract

We introduce and investigate the analytical properties of the periodic Evans function, which serves as a modern tool to explore the spectral stability of periodic traveling waves. The theory is then applied to the nonlinear Schrödinger (NLS) equation. We develop the theoretical framework for the periodic Evans function, extending its application from bounded domains to periodic systems via Floquet theory and Bloch-wave decomposition. Furthermore, monodromy matrix computations and winding number analysis, complement the theoretical results, providing a diverse approach to identifying spectral bands. This work bridges classical stability analysis with modern topological techniques, contributing to the understanding of nonlinear wave dynamics in dispersive systems.

Contents

Introduction

The early 20th century witnessed revolutionary ideas in stability analysis thanks to the development of dynamical systems theory. Aleksandr Lyapunov's seminal work on stability theory provided rigorous criteria for assessing whether perturbations grow or decay, using clever concepts like Lyapunov exponents. However, the chaotic complexity of nonlinear PDEs, such as those governing fluid dynamics and wave propagation, demanded new tools. The mid-20th century saw significant advances with the Hodgkin-Huxley model (1952), which described nerve impulse propagation as traveling waves, raising questions about their spectral stability. This biological context inspired John W. Evans, a mathematician with a medical background, to develop the Evans function in the 1970s. Initially aimed at proving the stability of nerve impulses, Evans' function unified functional analysis and dynamical systems by focusing the properties of the spectrum. His work, grounded in the Wronskian determinant methods and extended to unbounded domains, provided a modern framework for stability analysis, including applications to the NLS equation explored in this thesis.

In general, Nonlinear wave phenomena appear regularly in a wide array of physical systems, from the propagation of solitary water waves to the dynamics of nerve impulses and celestial structures. A central question in applied mathematics is the stability of such waves: assuming a small disturbance as our initial condition, the question is whether such small perturbations grow, leading to instability, or decay, preserving the wave's form. The solitary and periodic traveling wave solutions of the nonlinear Schrödinger equation (NLS), characterized by their oscillatory profiles, are of particular interest due to their relevance in fields such as quantum mechanics and nonlinear optics. This thesis investigates the spectral stability of these solutions.

Our study builds on the work of Kapitula and Promislow (2013), who provide a comprehensive framework for analyzing the spectral stability of nonlinear waves using the periodic Evans function. Their approach leverages Floquet theory and Bloch-wave decomposition to characterize the essential spectrum of linearized operators in periodic systems, such as the nonlinear Schrödinger equation, to which we apply the developed theory in the final chapter. The thesis further employs numerical techniques, including monodromy matrix computations and winding number analysis, to map spectral bands and confirm theoretical predictions. These topological methods, inspired by Kapitula and Promislow's emphasis on geometric and analytic tools, ensure a robust and integrative approach to understanding wave stability.

The thesis is structured as follows:

- **Chapter 1: Preliminaries** introduces essential concepts from linear functional analysis, ordinary differential equations (ODEs), and Floquet theory, laying the groundwork for stability analysis in infinite-dimensional spaces.
- **Chapter 2: The Evans Function** details the theoretical framework of the classical and periodic Evans functions, with examples illustrating their computation for Sturm-Liouville and Hill's equations, and introduces topological methods like winding number analysis. The culmination of this chapter will be application of the Evans function to the analysis of the spectral stability of both plane wave solutions of the nonlinear Schrödinger equation (NLS).

Chapter 1

Preliminaries

We begin by recalling some basic concepts in linear functional analysis, which play a central role in the stability theory of dynamical systems. To establish the necessity and connection of these topics with differential equations, we first provide a short introduction to some basic results in the theory of ODEs. Since the periodic Evans function serves as our main analytical tool, this chapter will subsequently focus further on Floquet theory. The majority of this chapter is based on standard references in the subject, in particular [10, 15, 2, 12, 14].

1.1 Some Known Results in The Theory of ODEs

We begin by recalling some results for systems of linear ODEs:

$$\frac{d}{dx}\boldsymbol{y} = \underbrace{\boldsymbol{A}(x)\boldsymbol{y}}_{f(x,\boldsymbol{y})} \text{ with the initial conditions: } \boldsymbol{y}(x_0) = \boldsymbol{y}_0 \tag{1.1}$$

Theorem 1.1.1 (Linearly Independent Solutions)**.** Let $\boldsymbol{A}(x) \in \mathbb{C}^{n\times n}$ be continuous on an open interval I. The complex-valued solutions of $\boldsymbol{y'} = \boldsymbol{A}(x)\boldsymbol{y}$ form a linear space V of dimension n.

The above theorem follows naturally from the principle of superposition. Recall how one often makes use of matrix functions $x \mapsto \boldsymbol{\Psi}(x)$, where each column of the $n \times n$ matrix $\boldsymbol{\Psi}(x)$ is a solution to (1.1). When choosing the column vectors, it is most practical to consider n-linearly independent solutions $\boldsymbol{y}_1, \dots, \boldsymbol{y}_n$ that span the solution space. In such case, we call $\boldsymbol{\Psi}(x)$ a *fundamental matrix solution (FMS)*. In addition, requiring that the FMS yields the identity matrix $\boldsymbol{I}$ at the starting point x_0 gives us the *Principal Fundamental Solution Matrix (Principal FMS)* at x_0.

$$\boldsymbol{\Psi}(x) := (\boldsymbol{y_1}, \dots, \boldsymbol{y_n})(x) \in \mathbb{C}^{n\times n}, \text{ subject to initial condition } \boldsymbol{\Psi}(x_0) = \boldsymbol{I}.$$

Remark 1.1.1. For any arbitrary choice of functions in $\boldsymbol{\Psi}$, the relation $\boldsymbol{\Phi}(x, x_0) := \boldsymbol{\Psi}(x)\boldsymbol{\Psi}^{-1}(x_0)$ holds. so that any matrix solution would be of the form: $\boldsymbol{\Psi}(x) = \boldsymbol{\Phi}(x, x_0)\boldsymbol{\Psi}(x_0)$ and hence the solution to (1.1) $\boldsymbol{y}(x) = \boldsymbol{\Phi}(x, x_0)\boldsymbol{y}(x_0)$

In some applied mathematics literature, particularly in control theory, the principal FMS is called "The State Transition Matrix", since it tracks the transition of the system from the initial starting point x_0. The state transition matrix provides a systematic way to analyze the stability and controllability of linear systems, determine solutions explicitly, and understand the time evolution of dynamic systems in engineering, physics, and applied mathematics.

Theorem 1.1.2. Let A be a linear mapping on a vector space $\mathcal{V}$ over $\mathbb{C}$. Then there are subspaces $\mathcal{V}_-, \mathcal{V}_0, \mathcal{V}_+$ which are invariant under A and such that:

a) $\mathcal{V} = \mathcal{V}_- \bigoplus \mathcal{V}_0 \bigoplus \mathcal{V}_+$.

b) all eigenvalues of A: $\mathcal{V}_- \to \mathcal{V}_-$ have a negative real part.

c) all eigenvalues of A: $\mathcal{V}_0 \to \mathcal{V}_0$ have the real part equal to zero.

d) all eigenvalues of A: $\mathcal{V}_+ \to \mathcal{V}_+$ have a positive real part.

This classification becomes of great use when studying *spectral stability* of dynamical systems.

1.2 Elements of Functional Analysis

Banach and Sobolev Spaces

Recall that every complete normed vector space constitutes a Banach space. Throughout this dissertation, we will extensively use and apply a number of already established results and theorems on Banach spaces. In particular, the focus would initially be on the basic Sobolev spaces, which combine properties of the L^p norm of a function and its (weak) derivatives up to a given order. One could formulate such notions in the following sense:

Definition 1.2.1 (The L^p Space)**.** Let $u : \mathbb{R} \to \mathbb{C}$ be Lebesgue measurable function For $p \geq 1$, the L^p space over $\mathbb{R}$ is defined as:

$$L^p(\mathbb{R}) = \left\{ u : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |u(x)|^p \, dx < \infty \right\}$$

With its associated norm being:

$$\|u\|_{L^p} = \|u\|_p = \left(\int_{\mathbb{R}} |u(x)|^p \, dx \right)^{1/p}.$$

Remark 1.2.1 (Special case: $p = 2$)**.** Recall that $L^2(\mathbb{R})$ is equipped with the following inner product:

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)} dx.$$

Recall that a function $u \in L^1_{\text{loc}}(\Omega)$ if, for every compact (closed and bounded) subset $K \subset \Omega$, the function u is integrable over K with respect to the Lebesgue measure. In other words,

$$\int_K |u(x)| \, dx < \infty$$

for every compact $K \subset \Omega$. In such cases we refer to u as a locally integrable function.

Definition 1.2.2 (Weak Derivative)**.** Let $u \in L^1_{\text{loc}}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open set. A function $v \in L^1_{\text{loc}}(\Omega)$ is called the *weak derivative* of u with respect to x_i (denoted $\frac{\partial u}{\partial x_i} = v$) if

$$\int_\Omega u(x) \frac{\partial \varphi(x)}{\partial x_i} \, dx = - \int_\Omega v(x)\varphi(x) \, dx$$

for all test functions $\varphi \in C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support in Ω.

Definition 1.2.3 (The $W^{k,p}$ Norm)**.** For k-times weakly differentiable functions $u : \mathbb{R} \to \mathbb{C}$ and $p \geq 1$, the $W^{k,p}$ norm is defined as:

$$\|u\|_{W^{k,p}(\mathbb{R})} = \left(\sum_{j=0}^{k} \| \frac{\partial^j}{\partial x} u\|^p_{L^p(\mathbb{R})} \right)^{1/p}$$

Remark 1.2.2. Unlike the L^p norm, which only accounts for the *size* of a function, the $W^{k,p}$ norm provides us with information on derivatives.

The above norm is associated with the Sobolev space $W^{k,p}(\mathbb{R})$, consisting of functions $u : \mathbb{R} \to \mathbb{C}$ with a finite $W^{k,p}$ norm,

$$W^{k,p}(\mathbb{R}) := \{u : ||u||_{W^{k,p}} < \infty\}.$$

Since the ultimate goal is to apply this theory to a practical setting, we fix $p = 2$, reducing $W^{k,p}(\mathbb{R})$ to the Hilbert space of square-integrable functions $L^2(\mathbb{R})$, and thereby avoiding unnecessary abstraction:

$$H^k := W^{k,2} \text{ and } H^0(\mathbb{R}) = L^2(\mathbb{R}).$$

Bounded and Closed Operators

Definition 1.2.4 (Dense Subset)**.** Let X be a Banach space. A subset $Y \subset X$ is said to be *dense* in X if for every $x \in X$, and for every $\epsilon > 0$, there exists $y \in Y$, such that $||x - y|| < \epsilon$

In more general terms, denseness in a Banach space means that one can approximate any element of the Banach space by elements of a dense subset of that space.

Remark 1.2.3. It is useful to think about a differential operator $\mathcal{L}$ as an unbounded linear operator on a Banach space X, which is only defined on a dense subspace $\mathcal{D}(\mathcal{L})$ of X. We call such an operator **densely defined**. Throughout the text, we assume that $\mathcal{L}$ is indeed densely defined.

Definition 1.2.5 (Closed Operator)**.** An operator $\mathcal{L}$ is **closed** if, whenever any sequence $\{u_j\} \subset D(\mathcal{L})$ converges in the norm of X to some $u \in X$, and the sequence $\{\mathcal{L}u_j\}$ converges to some $v \in X$, it follows that $u \in D(\mathcal{L})$ and $\mathcal{L}u = v$.

Definition 1.2.6 (Bounded Operator)**.** The operator $\mathcal{L} : Y \longrightarrow X$ is bounded from Y to X if the norm $||\mathcal{L}u||_X$ is uniformly bounded over the unit sphere:

$$\sup\{||\mathcal{L}u||_X : u \in Y, ||u||_Y = 1\} < \infty$$

Remark 1.2.4. Since operators act on elements and transform them, boundedness of $\mathcal{L}$ refers to a situation where the set of the transformed functions $\mathcal{L}v$ (the image) is bounded. In other words, there is a limit to the "strength" of $\mathcal{L}$ when it comes to transforming the functions. This interpretation only applies over bounded sets, allowing us to formulate the above definition by stating that a bounded operator maps bounded sets to bounded sets.

Definition 1.2.7. We denote the **space of bounded linear operators** from Y into X by $\mathcal{B}(Y, X)$. As for notation, if $Y = X$, we simply write $\mathcal{B}(X)$. Also note that $\mathcal{B}(Y, X)$ together with the following norm constitutes a Banach space.

$$||\mathcal{L}||_{\mathcal{B}(X,Y)} := \sup_{||u||_Y = 1} ||\mathcal{L}u||_X$$

Definition 1.2.8 (Compactness of $\mathcal{L}$)**.** If for each bounded sequence $\{u_j\} \subset Y$ the sequence $\{\mathcal{L}u_j\} \subset X$ has a convergent subsequence, then the operator $\mathcal{L}$ is said to be compact.

A compact operator is bounded. Furthermore, the sum of two compact operators is compact, and the composition of a compact operator and a bounded operator is compact.

Stability Through Spectral Analysis

Let $\mathcal{L}\colon D(\mathcal{L}) \subset X \to X$ be a closed and densely defined linear operator on a Banach space X.

Recall that the spectrum of $\mathcal{L}$ is a generalization of eigenvalues for operators on infinite-dimensional spaces. Formally, we have that:

Definition 1.2.9 (Spectrum)**.** The spectrum of a linear operator $\mathcal{L}$ is defined as:

$$\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} \,|\, (\mathcal{L} - \lambda\mathcal{I}) \text{ does not have a bounded inverse } \}.$$

An introductory understanding of how spectrum can help with stability analysis lies in the question of "where" in the spectrum the value λ lies. We have the following "regions":

Stable Spectrum: The Stable Spectrum consists of those values in the spectrum whose real parts are strictly negative. These correspond to modes that decay exponentially over time. In mathematical terms:

$$\sigma_{\text{stable}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) < 0\}$$

Center Spectrum: The Center Spectrum consists of values in the spectrum lying on the imaginary axis, corresponding to neutral or oscillatory dynamics. This set can be written as:

$$\sigma_{\text{center}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) = 0\}$$

Unstable Spectrum: The Unstable Spectrum consists of values in the spectrum whose real parts are strictly positive, corresponding to modes that grow exponentially over time. It is expressed as:

$$\sigma_{\text{unstable}}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) : \Re(\lambda) > 0\}$$

Definition 1.2.10 (Resolvent Set)**.** The resolvent set of $\mathcal{L}$, denoted by $\rho(\mathcal{L})$, is the set of all complex numbers $\lambda \in \mathbb{C}$ for which the operator $(\mathcal{L} - \lambda\mathcal{I})$ is invertible and the inverse $(\mathcal{L} - \lambda\mathcal{I})^{-1}$ is a bounded operator on X:

$$\rho(\mathcal{L}) = \left\{\lambda \in \mathbb{C} \,\middle|\, (\mathcal{L} - \lambda\mathcal{I}) \text{ is invertible (bijective) and } (\mathcal{L} - \lambda\mathcal{I})^{-1} \in \mathcal{B}(X)\right\}.$$

Definition 1.2.11 (Resolvent of $\mathcal{L}$)**.** The inverse operator $(\mathcal{L} - \lambda\mathcal{I})^{-1}$ is called the resolvent of $\mathcal{L}$.

Definition 1.2.12 (Adjoint Operator)**.** Let $\mathcal{L} : D(\mathcal{L}) \subset X \to Y$ be a densely defined linear operator between Hilbert spaces X and Y (i.e., $D(\mathcal{L})$ is dense in X). The adjoint $\mathcal{L}^* : D(\mathcal{L}^*) \subset Y \to X$ is defined such that for $y \in D(\mathcal{L}^*)$, the linear functional $x \mapsto \langle \mathcal{L}x, y\rangle_Y$ is continuous on $D(\mathcal{L})$. The domain $D(\mathcal{L}^*)$ consists of all $y \in Y$ for which there exists $z \in X$ such that:

$$\langle \mathcal{L}x, y\rangle_Y = \langle x, z\rangle_X \quad \forall x \in D(\mathcal{L}).$$

Then, $\mathcal{L}^*y = z$, and $D(\mathcal{L}^*)$ is the set of all such y.

Definition 1.2.13 (Fredholm Operator)**.** A linear operator $\mathcal{L} : X \to Y$ between Banach spaces X and Y is called a **Fredholm operator** if:

1. The kernel $\ker(\mathcal{L}) = \{x \in \mathcal{D}(X) \mid \mathcal{L}x = 0\}$ is finite-dimensional.
2. The range $\operatorname{ran}(\mathcal{L}) = \{\mathcal{L}x \mid x \in \mathcal{D}(X)\}$ is a closed subspace of Y.
3. The cokernel $\operatorname{coker}(\mathcal{L}) = Y/\operatorname{ran}(\mathcal{L})$ is finite-dimensional.

The **Fredholm index** of $\mathcal{L}$ is defined as:

$$\text{ind}(\mathcal{L}) = \dim(\ker(\mathcal{L})) - \dim(\text{coker}(\mathcal{L})).$$

In Hilbert spaces, the cokernel condition is equivalent to $\ker(\mathcal{L}^*) = \text{ran}(\mathcal{L})^\perp$ being finite-dimensional, where $\mathcal{L}^*$ is the adjoint operator.

This leads to the statement: An operator is Fredholm if and only if it has a finite Fredholm index.

At the beginning of this section, we portrayed the elements of the spectrum $\sigma(\mathcal{L})$ as complex numbers λ for which $(\mathcal{L} - \lambda I)$ is **not** invertible. In a finite-dimensional setting, this simply implies the existence of a non-trivial kernel. However, once we extend this concept to infinite-dimensional spaces, there are multiple situations where invertibility fails due to different reasons, and hence the elements λ of the spectrum would naturally partition the spectrum into subsets, each having implications for the stability analysis. At this stage of the discussion, we dichotomize the spectra into two categories:

1. The Point Spectrum σ_p
2. The Essential Spectrum σ_{ess}

Definition 1.2.14 (Spectrum of an Operator)**.** Let X be a Banach space and let $\mathcal{L} : D(\mathcal{L}) \subset X \to X$ be a closed linear operator with a dense domain $D(\mathcal{L})$ in X. The spectrum of $\mathcal{L}$ is partitioned into two sets:

(a) The **essential spectrum** of a Fredholm operator $\mathcal{L}$, denoted $\sigma_{\text{ess}}(\mathcal{L})$, consists of all $\lambda \in \mathbb{C}$ such that either:

- $\lambda I - \mathcal{L}$ is not Fredholm, or
- $\lambda I - \mathcal{L}$ is Fredholm, but $\text{ind}(\lambda I - \mathcal{L}) \neq 0$.

(b) The **point spectrum** of a Fredholm operator $\mathcal{L}$, denoted $\sigma_{\text{pt}}(\mathcal{L})$, is the set defined by:

$$\sigma_{\text{pt}}(\mathcal{L}) = \{\lambda \in \mathbb{C} : \text{ind}(\lambda I - \mathcal{L}) = 0, \text{ but } \lambda I - \mathcal{L} \text{ is not invertible}\}.$$

The elements of the point spectrum are called the **eigenvalues** of $\mathcal{L}$.

1.2.1 Floquet Theory

Motivating Question: Given a linear system of ordinary differential equations with periodic variable coefficients $A(x)$, will it follow from periodicity of A that the solution $\mathbf{y}(x)$ is also periodic? A Naive guess would simply suggest "yes". This, however, is generally not the case. We therefore seek methods to understand and analyze the solutions to periodic linear systems of the form:

$$\frac{d}{dx}\mathbf{y}(x) = \mathbf{A}(x)\mathbf{y}(x), \quad \mathbf{A}(x+T) \quad = \mathbf{A}(x) \tag{1.2}$$

1.2.2 Scalar Systems

In the case of constant matrices, it is well-known that the solutions are conveniently of the form $\mathbf{x}(t) = \exp(t\mathbf{A})\mathbf{x}_0$. In fact, if we consider all solutions, namely the Fundamental Matrix Solution (FMS) $\Phi(t)$, we can express Φ as an exponential of the matrix $\mathbf{A}$.

When it comes to variable matrices $\mathbf{A}(t)$, while existence and uniqueness theorems ensure the existence of the FMS, it soon becomes obvious that $\boldsymbol{\Phi}$ is generally *not* the exponential of any matrix. In the case of **periodic** matrices, we have an additional structure imposed on our system, making it possible to come up with an investigation of asymptotic behavior of solutions. This is the core idea of **Floquet Theory**, which exploits the mathematical properties of periodic matrices, and seeks to decompose $\mathbf{\Phi}$ to a periodic part, and an exponential of a **constant** matrix.

Without loss of generality, let us assume that the period is of length π. We begin by considering a time-dependent scalar problem

$$\dot{x}(t) = a(t)x(t), \quad a(t+\pi) \ = a(t) \tag{1.3}$$

It is assumed to be familiar to the reader that a general solution $\Phi(t)$ can be written in the form:

$$\Phi(t) = \exp\left(\int_0^t a(s)\, ds\right) \mathbf{x}_0$$

Define an average and a (net) deviation from the average

$$\bar{a} := \frac{1}{\pi}\int_0^\pi a(s)\, ds, \quad p(t) = \int_0^t (a(s) - \bar{a})\, ds. \tag{1.4}$$

Since $a(t)$ is periodic, the net deviation $p(t)$ from its mean is also periodic, since the behavior repeats itself in every cycle. Consider the exponential $P(t) = e^{p(t)}$. Together combined, we see that

$$\Phi(t) = P(t)e^{\bar{a}t}$$

1.2.3 Floquet Decomposition

A central result of Floquet theory is that the FMS $\Phi(x)$ can be factored into the product of a periodic matrix and an exponential term, provided that A is a periodic matrix:

$$\mathbf{\Phi}(x) = \mathbf{P}(x)e^{\mathbf{B}x}, \quad \mathbf{P}(x+T) = \mathbf{P}(x),$$

where $\mathbf{P}(x) \in \mathbb{C}^{n\times n}$ is periodic with period T, and $B \in \mathbb{C}^{n\times n}$ is a constant matrix, known as the *Floquet matrix.*

The eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of B are called the *Floquet exponents* and are related to the Floquet multipliers via

$$\rho_k = e^{\lambda_k T}, \quad k = 1, 2, \dots, n.$$

The decomposition implies that the solutions of the system can be written in the form

$$\mathbf{y}(x) = \mathbf{P}(x)e^{\mathbf{B}x}\mathbf{c},$$

where $\mathbf{c} \in \mathbb{C}^n$ is a constant vector. This representation separates the periodic oscillations from the exponential growth or decay, and essentially provides a canonical form for each fundamental matrix solution. The culmination of these efforts result in Floquet's theorem, which shows that there is a periodic, time-dependent change of coordinates that transforms the original periodic problem into a homogeneous linear system with constant coefficients.

Theorem 1.2.1 (Floquet Theorem)**.** Let $\mathbf{A}(x)$ be a continuous, T-periodic matrix-valued function, i.e., $\mathbf{A}(x+T) = \mathbf{A}(x)$ for all $x \in \mathbb{R}$, where $\mathbf{A}(x) : \mathbb{R} \to \mathbb{C}^{n\times n}$. Consider the linear system of ordinary differential equations:

$$\mathbf{y}' = \mathbf{A}(x)\mathbf{y}, \quad \mathbf{y} \in \mathbb{C}^n.$$

The Floquet Theorem states that there exists a fundamental solution matrix $\Phi(x)$ for this system such that:

$$\Phi(x) = \mathbf{P}(x)e^{\mathbf{B}x},$$

where:

1. $\mathbf{P}(x)$ is a continuous, T-periodic, invertible matrix-valued function, i.e., $\mathbf{P}(x+T) = \mathbf{P}(x)$,
2. $\mathbf{B}$ is a constant $n \times n$ matrix (possibly complex),
3. The monodromy matrix $\mathbf{M} = \Phi(T)$ has eigenvalues ρ_k, called **Floquet multipliers**, that determine the stability of solutions.

Furthermore, the solution $\mathbf{y} = \Phi(x)\mathbf{y}_0$ has Floquet multipliers given by the eigenvalues of M, and the characteristic exponents (or Floquet exponents) are the eigenvalues of B, satisfying $e^{T\mathbf{B}} = \mathbf{M}$.

1.2.4 Stability Analysis via Floquet Multipliers

The Floquet multipliers ρ_k determine the stability of the system:

If all $|\rho_k| < 1$, the corresponding solutions decay exponentially, and the system is stable.

If any $|\rho_k| > 1$, the corresponding solutions grow exponentially, and the system is unstable.

If $|\rho_k| < 1$ for all k but $|\rho_k| = 1$ for some k, the system is marginally stable, and the long-term behavior depends on higher-order terms.

Chapter 2

The Evans Function

2.0.1 Introduction

Since the late 19th century, numerous approaches have been developed for the stability analysis of nonlinear waves. One of the main approaches over the last few decades has been primarily based on a natural unification of ideas from functional analysis and dynamical systems. This fruitful effort has resulted in the development of the Evans function.

Introduced by J. Evans in the 1970s, the Evans function has since become a central concept in the stability analysis of nonlinear waves. In theory, we are discussing a complex-valued analytic function whose zeros correspond to eigenvalues of a linearized operator. It generalizes the Wronskian determinant and is particularly well-suited for boundary-value problems and eigenvalue problems on unbounded domains. The key advantage of the Evans function is its ability to encode spectral information, such as eigenvalue multiplicity and stability, in an analytically tractable form.

After a short introduction to the Evans function for the Sturm-Liouville problem, we will shift our focus on the **periodic** Evans function, which provides us with a rich framework for studying the stability of periodic traveling wave solutions to the Nonlinear Schrödinger equation (NLS). Its primary purpose is to identify and characterize the spectrum of the linearized operator about such waves, which is composed entirely of the essential spectrum. That is, the set of all complex numbers $\lambda \in \mathbb{C}$ for which the operator $(\mathcal{L} - \lambda I)$ fails to be Fredholm or when its Fredholm index is non-zero.

This chapter presents the theoretical framework, mathematical definitions, and properties of the Periodic Evans Function based on *Kapitula and Promislow's "Spectral and Dynamical Stability of Nonlinear Waves"* [10] and Gardner's foundational work [7].

2.1 Historical Origins: The Nerve Axon Equation

In mathematical biology, a nerve impulse is a signal that travels along a nerve, modeled as a solution to a partial differential equation. Checking whether small perturbations (disturbances) to the impulse grow (unstable) or decay (stable) over time motivates a thorough study of stability. The now-called "Evans function" appears for the first time in John W. Evans' 1975 paper, "Nerve Axon Equations: IV The Stable and the Unstable Impulse," published in the *Indiana University Mathematics Journal* (Vol. 24, No. 12, pp. 1169–1190). The paper models nerve impulses using a PDE that describes how a vector $W(x, t)$, representing voltage and other variables, evolves over time t and space x. The equation is:

$$\frac{\partial \boldsymbol{W}}{\partial t} = \begin{pmatrix} \frac{\partial^2 W^0}{\partial x^2} \\ 0 \\ \vdots \end{pmatrix} + g(W),$$

where $g(W)$ is a nonlinear function modeling nerve dynamics, and W^0 is the voltage component.

A nerve impulse is a traveling wave, $\boldsymbol{W}(x,t) = \phi(x - vt)$, where v is the wave speed, and $\phi(\xi)$ (with $\xi = x - vt$) is the wave profile. This profile satisfies an ordinary differential equation:

$$\begin{pmatrix} \frac{d^2\phi^0}{d\xi^2} \\ 0 \\ \vdots \end{pmatrix} + v\frac{d\phi}{d\xi} + g(\phi) = 0.$$

The answer to the question of stability depends on whether the linearized PDE around ϕ has solutions that grow exponentially. As a common approach, Evans perturbs the wave, $W = \phi + e^{\lambda t}\psi$, and linearizes the PDE, leading to an eigenvalue problem:

$$\begin{pmatrix} \frac{d^2\psi^0}{d\xi^2} \\ 0 \\ \vdots \end{pmatrix} + v\frac{d\psi}{d\xi} + (g_W(\phi) - \lambda I)\psi = 0,$$

Since linearization appears repeatedly throughout this thesis, we provide the following detailed computation that clarifies John W. Evans' conclusions:

2.1.1 The Equation in Detail

Essentially, we are studying a system of partial differential equations modeling nerve impulse propagation:

We have $\mathbf{W} = (W^0, W^1, \ldots, W^n)$, where W^0 is the voltage across the nerve membrane, and $W^1, \ldots, W^n$ are other variables (e.g., recovery variables in the FitzHugh-Nagumo system).

The first component includes diffusion, $\frac{\partial^2 W^0}{\partial x^2}$, modeling spatial spread of voltage.

The nonlinearity $g(\mathbf{W}) = (g^0(\mathbf{W}), g^1(\mathbf{W}), \ldots, g^n(\mathbf{W}))$ describes nerve dynamics (e.g., ionic currents).
Using the travelling wave ansatz $\mathbf{W}(x,t) = \phi(\xi)$ where $\xi = x - vt$, we make the following change of variables accordingly:

$$\frac{\partial \mathbf{W}}{\partial t} = \frac{d\boldsymbol{\phi}}{d\xi} \cdot \frac{\partial \xi}{\partial t} = -v\boldsymbol{\phi}'(\xi)$$

.

$$\frac{\partial^2 W^0}{\partial x^2} = \frac{d^2\phi^0}{d\xi^2}$$

.

The traveling wave equation therefore becomes:

$$\begin{pmatrix} \frac{d^2\phi^0}{d\xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v\boldsymbol{\phi}' + g(\boldsymbol{\phi}) = 0. \tag{2.1}$$

This ODE defines the nerve impulse profile $\boldsymbol{\phi}(\xi)$.

To study stability, we check if small disturbances to $\boldsymbol{\phi}(\xi)$ grow or decay. We perturb the solution and substitute $\mathbf{W}(x,t) = \boldsymbol{\phi}(\xi) + \epsilon\mathbf{U}(\xi,t)$ into the original PDE. The co-moving frame allows us to transform the derivatives using $\xi = x - vt$.

The time derivative is:

$$\frac{\partial \mathbf{W}}{\partial t} = \frac{\partial \mathbf{W}}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} + \left.\frac{\partial \mathbf{W}}{\partial t}\right|_{\text{explicit}}, \text{ where } \frac{\partial \xi}{\partial t} = -v.$$

$$\mathbf{W} = \boldsymbol{\phi}(\xi) + \epsilon\mathbf{U}(\xi,t), \quad \frac{\partial \mathbf{W}}{\partial \xi} = \boldsymbol{\phi}'(\xi) + \epsilon\frac{\partial \mathbf{U}}{\partial \xi}.$$

$$\left.\frac{\partial \mathbf{W}}{\partial t}\right|_{\text{explicit}} = \frac{\partial}{\partial t}[\boldsymbol{\phi}(\xi) + \epsilon\mathbf{U}(\xi,t)] = \epsilon\frac{\partial \mathbf{U}}{\partial t}.$$

$$\frac{\partial \mathbf{W}}{\partial t} = -v\boldsymbol{\phi}'(\xi) - \epsilon v\frac{\partial \mathbf{U}}{\partial \xi} + \epsilon\frac{\partial \mathbf{U}}{\partial t}. \tag{2.2}$$

$$\frac{\partial W^0}{\partial x} = \frac{\partial}{\partial x}[\phi^0(\xi) + \epsilon U^0(\xi,t)] = \frac{d\phi^0}{d\xi} + \epsilon\frac{\partial U^0}{\partial \xi},$$

$$\frac{\partial^2 W^0}{\partial x^2} = \frac{d^2\phi^0}{d\xi^2} + \epsilon\frac{\partial^2 U^0}{\partial \xi^2}.$$

$$\begin{pmatrix} \frac{\partial^2 W^0}{\partial x^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{d^2\phi^0}{d\xi^2} + \epsilon\frac{\partial^2 U^0}{\partial \xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

To evaluate $g(\mathbf{W}) = g(\boldsymbol{\phi} + \epsilon\mathbf{U})$, we apply Taylor expansion for small ϵ:

$$g(\boldsymbol{\phi} + \epsilon\mathbf{U}) = g(\boldsymbol{\phi}) + \epsilon g_{\mathbf{W}}(\boldsymbol{\phi})\mathbf{U} + \frac{\epsilon^2}{2}\sum_{i,j}\frac{\partial^2 g}{\partial W^i \partial W^j}(\boldsymbol{\phi})U^iU^j + O(\epsilon^3),$$

where $g_{\mathbf{W}}(\boldsymbol{\phi}) = \left(\frac{\partial g^i}{\partial W^j}\right)_{\mathbf{W}=\boldsymbol{\phi}}$ is the Jacobian matrix. For linearization, we keep terms up to $O(\epsilon)$:

$$g(\boldsymbol{\phi} + \epsilon\mathbf{U}) \approx g(\boldsymbol{\phi}) + \epsilon g_{\mathbf{W}}(\boldsymbol{\phi})\mathbf{U}.$$

Substitute into the original equation, we have:

$$-v\boldsymbol{\phi}'(\xi) - \epsilon v\frac{\partial \mathbf{U}}{\partial \xi} + \epsilon\frac{\partial \mathbf{U}}{\partial t} = \begin{pmatrix} \frac{d^2\phi^0}{d\xi^2} + \epsilon\frac{\partial^2 U^0}{\partial \xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + g(\boldsymbol{\phi}) + \epsilon g_{\mathbf{W}}(\boldsymbol{\phi})\mathbf{U} + O(\epsilon^2). \tag{2.3}$$

2.1.2 Linearizing the PDE

The traveling wave $\boldsymbol{\phi}(\xi)$ satisfies (2.1)

$$\begin{pmatrix} \dfrac{d^2\phi^0}{d\xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v\boldsymbol{\phi}' + g(\boldsymbol{\phi}) = 0.$$

Subtract this from (2.3) to eliminate terms involving only $\boldsymbol{\phi}$:

$$\begin{aligned}
&\left(-v\boldsymbol{\phi}'(\xi) - \epsilon v\frac{\partial \mathbf{U}}{\partial \xi} + \epsilon\frac{\partial \mathbf{U}}{\partial t}\right) - \left(-v\boldsymbol{\phi}'(\xi)\right) \\
&= \left(\begin{pmatrix} \dfrac{d^2\phi^0}{d\xi^2} + \epsilon\dfrac{\partial^2 U^0}{\partial \xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + g(\boldsymbol{\phi}) + \epsilon g_{\mathbf{W}}(\boldsymbol{\phi})\mathbf{U} + O(\epsilon^2)\right) \\
&\quad - \left(\begin{pmatrix} \dfrac{d^2\phi^0}{d\xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + g(\boldsymbol{\phi})\right).
\end{aligned} \tag{2.4}$$

$$\iff -\epsilon v\frac{\partial \mathbf{U}}{\partial \xi} + \epsilon\frac{\partial \mathbf{U}}{\partial t} = \begin{pmatrix} \epsilon\dfrac{\partial^2 U^0}{\partial \xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \epsilon g_{\mathbf{W}}(\boldsymbol{\phi})\mathbf{U} + O(\epsilon^2).$$

Divide by ϵ (since $\epsilon \neq 0$):

$$\frac{\partial \mathbf{U}}{\partial t} - v\frac{\partial \mathbf{U}}{\partial \xi} = \begin{pmatrix} \dfrac{\partial^2 U^0}{\partial \xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + g_{\mathbf{W}}(\boldsymbol{\phi})\mathbf{U} + O(\epsilon).$$

Neglect $O(\epsilon)$ terms for the linear approximation:

$$\frac{\partial \mathbf{U}}{\partial t} - v\frac{\partial \mathbf{U}}{\partial \xi} = \begin{pmatrix} \dfrac{\partial^2 U^0}{\partial \xi^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + g_{\mathbf{W}}(\boldsymbol{\phi})\mathbf{U}. \tag{2.5}$$

This is the linearized PDE governing the perturbation $\mathbf{U}(\xi, t)$.
Assume the pertrubation in exponential form:

$$\mathbf{U}(\xi, t) = e^{\lambda t}\boldsymbol{\psi}(\xi), \tag{2.6}$$

where $\boldsymbol{\psi}(\xi) = (\psi^0(\xi), \psi^1(\xi), \ldots, \psi^n(\xi))$, and $\lambda \in \mathbb{C}$

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial}{\partial t}[e^{\lambda t}\boldsymbol{\psi}(\xi)] = \lambda e^{\lambda t}\boldsymbol{\psi}(\xi),$$

$$\frac{\partial \mathbf{U}}{\partial \xi} = \frac{\partial}{\partial \xi}[e^{\lambda t}\boldsymbol{\psi}(\xi)] = e^{\lambda t}\boldsymbol{\psi}'(\xi),$$

$$\frac{\partial^2 U^0}{\partial \xi^2} = \frac{\partial^2}{\partial \xi^2}[e^{\lambda t}\psi^0(\xi)] = e^{\lambda t}\psi''^0(\xi).$$

Substitute into (2.4)

$$\lambda e^{\lambda t}\boldsymbol{\psi}(\xi) - v e^{\lambda t}\boldsymbol{\psi}'(\xi) = e^{\lambda t}\begin{pmatrix}\psi''^0(\xi)\\ 0\\ \vdots\\ 0\end{pmatrix} + e^{\lambda t} g_{\mathbf{W}}(\boldsymbol{\phi})\boldsymbol{\psi}(\xi).$$

$$\lambda\boldsymbol{\psi}(\xi) - v\boldsymbol{\psi}'(\xi) = \begin{pmatrix}\psi''^0(\xi)\\ 0\\ \vdots\\ 0\end{pmatrix} + g_{\mathbf{W}}(\boldsymbol{\phi})\boldsymbol{\psi}(\xi).$$

$$\begin{pmatrix}\psi''^0(\xi)\\ 0\\ \vdots\\ 0\end{pmatrix} + v\boldsymbol{\psi}'(\xi) + [g_{\mathbf{W}}(\boldsymbol{\phi}) - \lambda I]\boldsymbol{\psi}(\xi) = 0. \tag{2.7}$$

This gives us the eigenvalue problem (see paper, p. 1171), with the operator

$$L\boldsymbol{\psi} = \begin{pmatrix}\psi''^0\\ 0\\ \vdots\\ 0\end{pmatrix} + v\boldsymbol{\psi}' + g_{\mathbf{W}}(\boldsymbol{\phi})\boldsymbol{\psi}.$$

Stability requires no bounded solutions $\boldsymbol{\psi}(\xi)$ (i.e., $\boldsymbol{\psi}(\xi) \to 0$ as $|\xi| \to \infty$) for $\mathrm{Re}\lambda \geq 0$, except possibly at $\lambda = 0$ (which requires further analysis).

The main innovation in the paper is the *Evans function*, $E(\lambda)$, a complex-valued function that helps find eigenvalues λ. We provide a brief summary of its outline before discussing it in detail.

- The eigenvalue problem is rewritten as a first-order ODE system:

  $$\frac{d\tilde{\psi}}{d\xi} = (\tilde{A}_\lambda + \tilde{P}(\xi))\tilde{\psi},$$

  where $\tilde{\psi}$ includes ψ and its derivatives, $\tilde{A}_\lambda$ is a constant matrix, and $\tilde{P}(\xi)$ depends on the wave profile.

- Evans identifies solutions to this ODE that decay as $\xi \to +\infty$ (originally he denotes these as $\tilde{\beta}(\lambda, \xi)$) and solutions to the adjoint system that decay as $\xi \to -\infty$ (called $\beta^*(\lambda, y)$).

- The Evans function is defined as:

$$E(\lambda) = {}^t\beta^*(\lambda,\xi)\tilde{\beta}(\lambda,\xi),$$

which is a scalar product of these solutions, independent of ξ. If $E(\lambda) = 0$, it means $\tilde{\beta}$ is bounded as $\xi \to -\infty$, indicating an eigenvalue λ. Note that this is because the scalar product ${}^t\beta^*(\lambda,\xi) \cdot \tilde{\beta}(\lambda,\xi)$ is independent of ξ. If $E(\lambda) = 0$, then $\tilde{\beta}(\lambda,\xi)$ is orthogonal to $\beta^*(\lambda,\xi)$ for all ξ. Since $\beta^*(\lambda,\xi)$ spans the stable subspace of the adjoint system at $\xi \to -\infty$, this orthogonality implies that $\tilde{\beta}(\lambda,\xi)$ has no component in the unstable subspace of the original system at $\xi \to -\infty$. Consequently, $\tilde{\beta}(\lambda,\xi)$ lies in the stable subspace at $\xi \to -\infty$, ensuring it is bounded (or decays) as $\xi \to -\infty$. Thus, $\tilde{\beta}(\lambda,\xi)$ is a bounded eigenfunction of $\mathcal{L}$, and λ is an eigenvalue.

- The function $E(\lambda)$ is analytic (smooth in the complex plane), and its zeros correspond to eigenvalues. The order of a zero matches the eigenvalue's multiplicity.

The 1972 paper focuses on the FitzHugh-Nagumo system, meaning that Evans function's general applicability is not fully explored. Relying on later works by Jones, Alexander, Kapitula, Promislow and Gardner, we explore the extension of this theory to *periodic waves* and their applications in hydrodynamics.

2.2 The Classical Evans Function

In the stability analysis of the Nerve Axon equation, as explored by Evans in his 1970s papers, traveling wave solutions are typically studied on the unbounded domain $\mathbb{R}$, where the Evans function detects eigenvalues of the linearized operator, often involving a continuous spectrum. To introduce a simpler, complementary framework, we consider Sturm-Liouville operators on the bounded interval $[-1,1]$, which yield a discrete spectrum and serve as a foundational example for eigenvalue problems in stability analysis. A Sturm-Liouville operator is defined as:

$$L(p) := \frac{d^2p}{dx^2} + a_1(x)\frac{dp}{dx} + a_0(x)p,$$

where $a_0, a_1 \in C[-1,1]$ are real-valued. The associated boundary value problem is:

$$Lp = \lambda p, \quad x \in [-1,1],$$

with separated boundary conditions. This setting facilitates the study of discrete eigenvalues, providing insight into stability properties before addressing unbounded domains with periodic coefficients, as in the Nonlinear Schrödinger Equation.

The following illustrates a compact way to write the separated boundary conditions:

$$\begin{pmatrix} b_1^- & b_2^- \end{pmatrix} \begin{pmatrix} p \\ \partial_x p \end{pmatrix}(-1) = 0. \tag{2.8}$$

$$\begin{pmatrix} b_1^+ & b_2^+ \end{pmatrix} \begin{pmatrix} p \\ \partial_x p \end{pmatrix}(+1) = 0. \tag{2.9}$$

We introduce boundary vectors

$$\boldsymbol{b}^{a\pm} := \begin{pmatrix} b_1^\pm & b_2^\pm \end{pmatrix} \text{ with the norm } ||\boldsymbol{b}^{a\pm}|| = 1$$

Remark 2.2.1. The above restriction on the norm is a convention to ensure that the boundary conditions depend on the direction of the newly introduced $\boldsymbol{b}^{a\pm}$ and not its magnitude.

2.2.1 The Construction of The Evans Function

Jost Functions & Scattering Theory of Quantum Mechanics

In quantum mechanics, studying how waves behave after having interacted with an object or a region has provided us with an abundance of rich mathematical theories. Akin to classical mechanics, where a ball might bounce back, pass through, or change direction after having been thrown at a glass pane, one could consider waves as particle-like entities, and thereby classify their behavior before and after the interaction. Mainly, we investigate waves *coming in toward* the obstacle (incoming waves) and waves *moving away* after interaction (outgoing wave) (see figure 2.1. Considering these as two types of solutions to a differential equation, we see that studying their eigenvalues reveals much about the stability and energy levels. The construction of the Evans Function is motivated by its ability to detect eigenvalues in stability analyses of nonlinear waves, as detailed in [10].

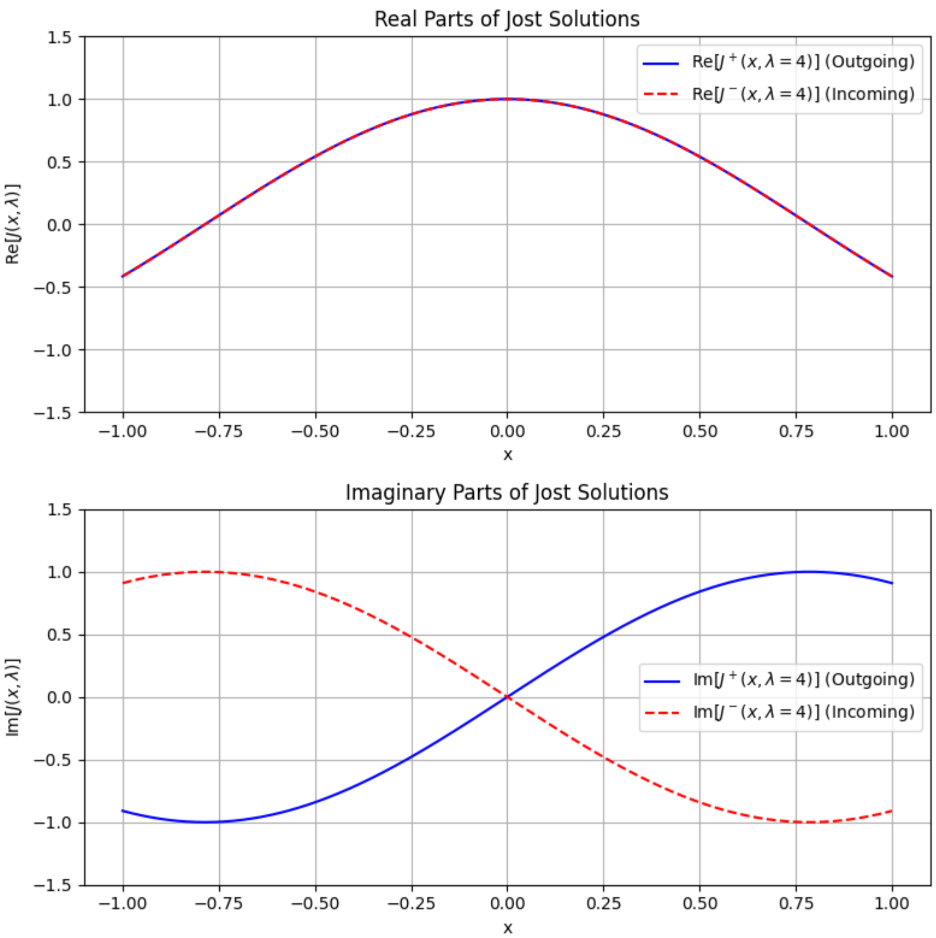

Figure 2.1: Real and imaginary parts of Jost solutions $J^+(x, \lambda = 4)$ (outgoing, blue) and $J^-(x, \lambda = 4)$ (incoming, red dashed) for the free particle Schrödinger equation, illustrating their oscillatory behavior analogous to incoming and outgoing waves in scattering theory [13, 11].

This approach draws parallels with scattering theory, where the so-called *Jost solutions* $\boldsymbol{J}^+(x, \lambda)$ and $\boldsymbol{J}^-(x, \lambda)$ act as incoming and outgoing waves. For some specific values of λ, the behavior of the system changes, which could indicate instability. This is particularly useful in wave propagation problems [13]. In a purely mathematical context, one can borrow Jost solutions from scattering theory to capture those values of λ that result in a collapse of linear independence in the solution space. Those are indeed our eigenvalues.

We begin by rewriting the eigenvalue problem (2.2) as a dynamical system:

$$\text{Let } \boldsymbol{Y} := \begin{bmatrix} p(x) \\ p'(x) \end{bmatrix} \text{ so that we have } \boldsymbol{Y}' = \boldsymbol{A}(x,\lambda)\boldsymbol{Y}$$

$$\text{where } \boldsymbol{A}(x,\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - a_0(x) & -a_1(x) \end{pmatrix}$$

Instead of solving explicitly for $\boldsymbol{Y}(x)$, we intend to understand the geometric structure of the solution space.

Definition 2.2.1 (Boundary Subspaces)**.** Let $\boldsymbol{b}^{\pm} := (-b_2^{\pm}, b_1^{\pm})^{\intercal}$. The boundary space is defined as:

$$\mathbb{B}_{\pm} = \text{ span } \{\boldsymbol{b}^{\pm}\}$$

From the theory of ordinary differential equations, we have that there exist 2 linearly independent solutions $\boldsymbol{Y}_1, \boldsymbol{Y}_2$ that satisfy the dynamical system. Using the notation discussed in chapter 1 for the principal FMS, we introduce the *Jost* solutions:

$$\mathbf{J}^{\pm}(x,\lambda) = \Psi(x,\lambda)\Psi^{-1}(\pm 1,\lambda)\boldsymbol{b}^{\pm}$$

$$\mathbf{J}^{\pm}(\pm 1,\lambda) = \boldsymbol{b}^{\pm} \text{ when the domain of } x \text{ is bounded.}$$

We can therefore write every other solution as a linear combination of $\mathbf{J}^{+}$ and $\mathbf{J}^{-}$, provided that they are linearly independent.

Jost solutions are constructed to satisfy specific boundary conditions at $x = \pm 1$, by which the behavior of solutions are highlighted as they approach the boundaries, much like in scattering theory where they describe incoming and outgoing waves [11]. This makes them ideal for detecting eigenvalues, as their linear dependence at specific λ values indicates an eigenvalue. Unlike Green's functions, which are used to solve inhomogeneous differential equations by representing the response to a point source [4], Jost solutions focus on the homogeneous problem and the asymptotic behavior of solutions.

Remark 2.2.2 (Detecting Eigenvalues via The Evans Function)**.** If $\lambda = \lambda_0$ is an eigenvalue, then the linear independency breaks, and hence the spanned system collapses. That is, at an eigenvalue, the Jost solutions coincide up to a scalar multiple.

Definition 2.2.2 (The Evans Matrix)**.** We consider an 2×2 complex-valued matrix $\boldsymbol{E}(\lambda)$ of the spectral parameter λ that has columns $\boldsymbol{J}^{\pm}(0,\lambda)$:

$$\boldsymbol{E}(\lambda) = \begin{bmatrix} | & | \\ \boldsymbol{J}^{-} & \boldsymbol{J}^{+} \\ | & | \end{bmatrix}(0,\lambda)$$

We are now finally well-equipped to introduce the Evans function.

Definition 2.2.3 (The Evans Function)**.** Taking the determinant of the Evans matrix, we will have the Evans function, which is the Wronskian of the specific solutions $\mathbf{J}^{\pm}$.

$$E(\lambda) = \det \boldsymbol{E}(\lambda)$$

Example 2.2.1 (A Constant-Coefficient Sturm-Liouville Problem)**.** Consider the Sturm-Liouville problem with constant coefficients on the interval $[-1,1]$:

$$-\frac{d^2p}{dx^2} = \lambda p, \quad x \in [-1,1],$$

with Dirichlet boundary conditions $p(-1) = 0$, $p(1) = 0$. This is equivalent to the eigenvalue problem $Lp = \lambda p$, where $L = -\dfrac{d^2}{dx^2}$, and we seek eigenvalues λ for which a non-trivial solution $p(x)$ satisfies the boundary conditions.

Step 1: Rewrite as a First-Order System. We begin by rewriting the second-order ODE as a first-order system. Let the vector $\tilde{p} := \begin{pmatrix} p \\ p' \end{pmatrix}$. Differentiating, we get:

$$\frac{d\tilde{p}}{dx} = \begin{pmatrix} p' \\ p'' \end{pmatrix} = \begin{pmatrix} p' \\ -\lambda p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} \tilde{p} = A_\lambda \tilde{p},$$

where $A_\lambda = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}$ is a constant matrix.

Step 2: Find Fundamental Solutions. The system $\dfrac{d\tilde{p}}{dx} = A_\lambda \tilde{p}$ has solutions determined by the eigenvalues of A_λ. The characteristic equation is:

$$\det(A_\lambda - \mu I) = \det \begin{pmatrix} -\mu & 1 \\ -\lambda & -\mu \end{pmatrix} = \mu^2 + \lambda = 0, \quad \mu = \pm\sqrt{-\lambda}.$$

Let $\mu = \sqrt{-\lambda}$, choosing the principal square root (e.g., $\mathrm{Re}(\mu) \geq 0$ or $\mathrm{Im}(\mu) \geq 0$). The general solution to the ODE is:

$$p(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} = c_1 \cosh(\mu x) + c_2 \sinh(\mu x).$$

Corresponding solutions to the system are:

$$\tilde{p}_1(x) = \begin{pmatrix} e^{\mu x} \\ \mu e^{\mu x} \end{pmatrix}, \quad \tilde{p}_2(x) = \begin{pmatrix} e^{-\mu x} \\ -\mu e^{-\mu x} \end{pmatrix}.$$

The fundamental solution matrix is:

$$\Phi(x, \lambda) = \begin{pmatrix} e^{\mu x} & e^{-\mu x} \\ \mu e^{\mu x} & -\mu e^{-\mu x} \end{pmatrix}.$$

The Wronskian of these solutions is:

$$W(x) = \det \Phi(x, \lambda) = e^{\mu x} \cdot (-\mu e^{-\mu x}) - e^{-\mu x} \cdot (\mu e^{\mu x}) = -\mu e^{\mu x - \mu x} - \mu e^{-\mu x + \mu x} = -2\mu,$$

which is constant, as expected since $\mathrm{tr}(A_\lambda) = 0$.

Step 3: Solutions Satisfying Boundary Conditions. For the boundary value problem, we need solutions satisfying the Dirichlet conditions:

- $p^-(-1) = 0$: Let $p^-(x) = c \sinh(\mu(x+1))$, since $\sinh(\mu(-1+1)) = \sinh(0) = 0$. Thus:

$$\tilde{p}^-(x) = \begin{pmatrix} \sinh(\mu(x+1)) \\ \mu \cosh(\mu(x+1)) \end{pmatrix}.$$

- $p^+(1) = 0$: Let $p^+(x) = c \sinh(\mu(x-1))$, since $\sinh(\mu(1-1)) = \sinh(0) = 0$. Thus:

$$\tilde{p}^+(x) = \begin{pmatrix} \sinh(\mu(x-1)) \\ \mu \cosh(\mu(x-1)) \end{pmatrix}.$$

Step 4: Define the Evans Function. The Evans function is the Wronskian of the solutions $\tilde{p}^-(x)$ and $\tilde{p}^+(x)$, evaluated at a point, say $x = 0$:

$$\begin{aligned}
E(\lambda) &= \det \begin{pmatrix} \tilde{p}^-(0) & \tilde{p}^+(0) \end{pmatrix} \\
&= \det \begin{pmatrix} \sinh\big(\mu(0+1)\big) & \sinh\big(\mu(0-1)\big) \\ \mu\cosh\big(\mu(0+1)\big) & \mu\cosh\big(\mu(0-1)\big) \end{pmatrix} \\
&= \det \begin{pmatrix} \sinh(\mu) & \sinh(-\mu) \\ \mu\cosh(\mu) & \mu\cosh(-\mu) \end{pmatrix}.
\end{aligned}$$

Since $\sinh(-\mu) = -\sinh(\mu)$ and $\cosh(-\mu) = \cosh(\mu)$, the matrix becomes:

$$\begin{pmatrix} \sinh(\mu) & -\sinh(\mu) \\ \mu\cosh(\mu) & \mu\cosh(\mu) \end{pmatrix}.$$

Computing the determinant, we will have:

$$\begin{aligned}
E(\lambda) &= \sinh(\mu) \cdot \mu\cosh(\mu) - \big(-\sinh(\mu)\big) \cdot \mu\cosh(\mu) \\
&= \mu\cosh(\mu)\sinh(\mu) + \mu\cosh(\mu)\sinh(\mu) \\
&= 2\mu\cosh(\mu)\sinh(\mu) \\
&= \mu\left[2\sinh(\mu)\cosh(\mu)\right] \\
&= \mu\ \sinh(2\mu).
\end{aligned}$$

Step 5: Find Eigenvalues. The Evans function vanishes when:

$$E(\lambda) = \mu\sinh(2\mu) = 0.$$

This occurs when:

- $\mu = 0$, so $\lambda = -\mu^2 = 0$. For $\lambda = 0$, the ODE is $-\frac{d^2p}{dx^2} = 0$, with solution $p(x) = ax + b$. Applying $p(-1) = 0$, $p(1) = 0$:

  $$p(-1) = -a + b = 0, \quad p(1) = a + b = 0 \implies a = 0, b = 0 \implies p(x) = 0.$$

  Thus, $\lambda = 0$ is not an eigenvalue (no non-trivial solution).

- $\sinh(2\mu) = 0$, so $2\mu = ik\pi$, $k \in \mathbb{Z}$, hence $\mu = ik\pi/2$. Then:

  $$\lambda = -\mu^2 = -\left(\frac{ik\pi}{2}\right)^2 = \left(\frac{k\pi}{2}\right)^2, \quad k = 1, 2, \ldots.$$

For $\lambda_k = \left(\frac{k\pi}{2}\right)^2$, the solution is:

$$p(x) = c_1\cos\left(\frac{k\pi x}{2}\right) + c_2\sin\left(\frac{k\pi x}{2}\right).$$

Apply boundary conditions:

$$p(-1) = c_1\cos\left(\frac{k\pi}{2}\right) + c_2\sin\left(\frac{k\pi}{2}\right) = 0, \quad p(1) = c_1\cos\left(\frac{k\pi}{2}\right) - c_2\sin\left(\frac{k\pi}{2}\right) = 0.$$

Since $\cos(k\pi/2) = 0$ and $\sin(k\pi/2) = \pm 1$ for odd k, or vice versa for even k, we get non-trivial solutions $p(x) = \sin(k\pi x/2)$, satisfying both conditions. Thus, the eigenvalues are $\lambda_k = \left(\frac{k\pi}{2}\right)^2$, $k = 1, 2, \ldots$.

2.3 The Periodic Evans Function

We now focus on the spectral stability of periodic traveling wave solutions to higher-order dispersive equations, such as the NLS equation. Our goal is to extend the Evans function framework, previously developed for non-periodic systems, to periodic settings and show that its roots correspond to eigenvalues of the linearized operator.

Consider the linearized operator obtained from linearizing about a periodic traveling wave (or heteroclinic/homoclinic equilibrium):

$$\mathcal{L}p = \frac{d^n}{dx^n}p + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}p + \cdots + a_1(x)\frac{d}{dx}p + a_0(x)p, \tag{2.10}$$

where $p \in H^n(\mathbb{R})$, $n \geq 1$, and the coefficients $a_j(x)$ are smooth and T-periodic, i.e., $a_j(x+T) = a_j(x)$ for $j = 0, \ldots, n-1$. Note that this operator $\mathcal{L} : H^n(\mathbb{R}) \to L^2(\mathbb{R})$ is defined on the whole real line, where the periodicity of the coefficients leads to a spectrum that is purely essential (essential, often consisting of bands). This contrasts with non-periodic boundary value problems (BVPs) with separated boundary conditions on finite intervals, which typically have a purely point spectrum. However, if we restrict to seeking T-periodic solutions p (i.e., $p(x+T) = p(x)$), the problem reduces to an eigenvalue problem on the finite interval $[0, T]$ with periodic boundary conditions, yielding a purely point spectrum, similar to the case of separated boundary conditions on finite intervals. This is where Floquet theory plays a helpful role, and effectively reduces the analysis on the whole line to a family of such periodic problems parameterized by the Floquet multiplier (or Bloch parameter) $\mu \in [0, 2\pi/T)$, with the full spectrum obtained as the union over all μ.

To analyze the eigenvalue problem $\mathcal{L}p = \lambda p$, we rewrite (2.10) as a first-order system. Define the state vector $\mathbf{Y} = (p, p', \ldots, p^{(n-1)})^\top$. The eigenvalue problem becomes:

$$\mathbf{Y}' = \mathbf{A}(x, \lambda)\mathbf{Y}, \tag{2.11}$$

where $\mathbf{A}(x, \lambda)$ is the $n \times n$ matrix:

$$\mathbf{A}(x, \lambda) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \lambda - a_0(x) & -a_1(x) & \cdots & -a_{n-2}(x) & -a_{n-1}(x) \end{bmatrix}, \tag{2.12}$$

with $\mathbf{A}(x+T, \lambda) = \mathbf{A}(x, \lambda)$ due to the periodicity of the coefficients.

2.3.1 Floquet and Bloch-Wave Decomposition

The periodicity of $\mathbf{A}(x, \lambda)$ allows us to apply Floquet theory. Every fundamental matrix solution $\Phi(x, \lambda)$ of (2.11) can be written as:

$$\Phi(x, \lambda) = \mathbf{P}(x, \lambda)e^{\mathbf{B}(\lambda)x}, \tag{2.13}$$

where $\mathbf{P}(x, \lambda)$ is T-periodic ($\mathbf{P}(x+T, \lambda) = \mathbf{P}(x, \lambda)$) and $\mathbf{B}(\lambda)$ is a constant matrix. For the eigenvalue problem $\mathcal{L}p = \lambda p$ to have a non-trivial solution $p \in H^n(\mathbb{R})$, the solution must be bounded and decay at infinity. However, the Floquet representation (2.13) implies that solutions are of the form $p(x) = \mathbf{P}(x, \lambda)e^{\mathbf{B}(\lambda)x}$, where $\mathbf{P}(x, \lambda)$ is T-periodic. Here, the point spectrum of $\mathcal{L}$ on $\mathbb{R}$ is empty, and the spectrum is purely essential, consisting of λ for which there exist bounded solutions (see [10, pp. 68]).

To characterize the essential spectrum of $\mathcal{L}$, we use a Bloch-wave decomposition. Assume solutions of the form:

$$p(x) = e^{i\mu x}q(x), \quad \mu \in [0, 2\pi/T), \tag{2.14}$$

where $q(x)$ is T-periodic. Substituting into $\mathcal{L}p = \lambda p$, the eigenvalue problem becomes:

$$\mathcal{L}_\mu q = \left(\left(\frac{d}{dx} + i\mu \right)^n + a_{n-1}(x) \left(\frac{d}{dx} + i\mu \right)^{n-1} + \cdots + a_1(x) \left(\frac{d}{dx} + i\mu \right) + a_0(x) \right) q = \lambda q, \tag{2.15}$$

where $\mathcal{L}_\mu$ acts on T-periodic functions in $H^n([0,T])$ with periodic boundary conditions. The operator $\mathcal{L}_\mu$ has a discrete point spectrum, and the essential spectrum of $\mathcal{L}$ consists of those λ for which there exists a bounded solution of the form (2.14), where μ is a Floquet exponent of $\mathbf{A}(x, \lambda)$. Specifically, the essential spectrum is given by:

$$\lambda \in \sigma_{\text{ess}}(\mathcal{L}) \iff \lambda \in \sigma_{\text{pt}}(\mathcal{L}_\mu) \text{ for some } \mu \in [0, 2\pi/T), \tag{2.16}$$

as shown in [10, pp. 68]. This decomposition reduces the analysis on the whole line to a family of eigenvalue problems on $[0, T]$, with the full spectrum obtained as the union over all μ.

Converting (2.15) to a first-order system, let $\tilde{\mathbf{Y}} = (q, q', \ldots, q^{(n-1)})^\top$. The system becomes:

$$\tilde{\mathbf{Y}}' = (\mathbf{A}(x, \lambda) - i\mu \mathbf{I}_n)\tilde{\mathbf{Y}}, \tag{2.17}$$

with periodic boundary conditions $\tilde{\mathbf{Y}}(0, \lambda) = \tilde{\mathbf{Y}}(T, \lambda)$. The original solution is $\mathbf{Y}(x) = e^{i\mu x}\tilde{\mathbf{Y}}(x)$, satisfying:

$$\mathbf{Y}(T, \lambda, \mu) = e^{i\mu T}\mathbf{Y}(0, \lambda, \mu). \tag{2.18}$$

This non-separated boundary condition complicates the direct application of the classical Evans function, which assumes separated boundary conditions.

2.3.2 Extended System for the Periodic Evans Function

To define a periodic Evans function, we embed the system in a larger space to handle the periodic boundary conditions. Define an extended vector:

$$\mathbf{W} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}, \tag{2.19}$$

where $\mathbf{Z}' = \mathbf{0}$, and $\mathbf{Z}$ is an auxiliary variable to enforce boundary conditions. The extended system is:

$$\mathbf{W}' = \begin{bmatrix} \mathbf{A}(x, \lambda, \mu) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{W}, \tag{2.20}$$

where $\mathbf{A}(x, \lambda, \mu) = \mathbf{A}(x, \lambda) - i\mu \mathbf{I}_n$. The boundary conditions for $\mathbf{Y}$ are enforced via:

$$\mathbf{Y}(0) = \mathbf{Z}(0), \quad \mathbf{Y}(T) = \mathbf{Z}(T). \tag{2.21}$$

The following theorem ensures equivalence to the periodic boundary condition:

Lemma 2.3.1. Let $\mathbf{W}$ be as defined above, and consider the system $\mathbf{Y}' = \mathbf{A}(x, \lambda, \mu)\mathbf{Y}$ with $\mathbf{Y}(0) = \mathbf{Y}(T)$. Then $\mathbf{W}$ solves (2.20) with boundary conditions $\mathbf{Y}(0) = \mathbf{Z}(0)$, $\mathbf{Y}(T) = \mathbf{Z}(T)$ if and only if $\mathbf{Y}$ satisfies $\mathbf{Y}(0) = \mathbf{Y}(T)$.

Proof. We show that a solution $\mathbf{W} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}$ to the extended system (2.20) with boundary conditions $\mathbf{Y}(0) = \mathbf{Z}(0)$, $\mathbf{Y}(T) = \mathbf{Z}(T)$ implies that $\mathbf{Y}$ satisfies $\mathbf{Y}' = \mathbf{A}(x, \lambda, \mu)\mathbf{Y}$ with $\mathbf{Y}(0) = \mathbf{Y}(T)$, and conversely, that a solution $\mathbf{Y}$ to the latter with periodic boundary conditions can be extended to a solution $\mathbf{W}$ satisfying the extended system and boundary conditions.

Forward Direction. ($\Longrightarrow$) Assume $\mathbf{W} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}$ solves the extended system:

$$\mathbf{W}' = \begin{bmatrix} \mathbf{A}(x, \lambda, \mu) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{W},$$

with boundary conditions $\mathbf{Y}(0) = \mathbf{Z}(0)$, $\mathbf{Y}(T) = \mathbf{Z}(T)$. The system implies:

$$\mathbf{W}' = \begin{pmatrix} \mathbf{Y}' \\ \mathbf{Z}' \end{pmatrix} = \begin{pmatrix} \mathbf{A}(x, \lambda, \mu)\mathbf{Y} \\ \mathbf{0} \end{pmatrix}.$$

Thus, $\mathbf{Y}' = \mathbf{A}(x, \lambda, \mu)\mathbf{Y}$, so $\mathbf{Y}$ satisfies the original system, and $\mathbf{Z}' = \mathbf{0}$, implying that $\mathbf{Z}(x) = \mathbf{C}$, a constant vector in $\mathbb{C}^n$. The boundary conditions give:

$$\mathbf{Y}(0) = \mathbf{Z}(0) = \mathbf{C}, \quad \mathbf{Y}(T) = \mathbf{Z}(T) = \mathbf{C}.$$

Since $\mathbf{Z}(0) = \mathbf{Z}(T) = \mathbf{C}$, it follows that:

$$\mathbf{Y}(0) = \mathbf{Y}(T) = \mathbf{C}.$$

Thus, $\mathbf{Y}$ satisfies the periodic boundary condition $\mathbf{Y}(0) = \mathbf{Y}(T)$.

Reverse Direction. ($\Longleftarrow$) Now assume $\mathbf{Y}$ satisfies $\mathbf{Y}' = \mathbf{A}(x, \lambda, \mu)\mathbf{Y}$ with the periodic boundary condition $\mathbf{Y}(0) = \mathbf{Y}(T)$. Define a constant vector $\mathbf{Z}(x) = \mathbf{Y}(0)$, so $\mathbf{Z}' = \mathbf{0}$. Construct the extended vector:

$$\mathbf{W} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Y}(0) \end{pmatrix}.$$

We verify that $\mathbf{W}$ solves the extended system:

$$\mathbf{W}' = \begin{pmatrix} \mathbf{Y}' \\ \mathbf{Z}' \end{pmatrix} = \begin{pmatrix} \mathbf{A}(x, \lambda, \mu)\mathbf{Y} \\ \mathbf{0} \end{pmatrix} = \begin{bmatrix} \mathbf{A}(x, \lambda, \mu) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{Y} \\ \mathbf{Y}(0) \end{pmatrix} = \begin{bmatrix} \mathbf{A}(x, \lambda, \mu) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{W}.$$

Next, check the boundary conditions:

- At $x = 0$: $\mathbf{Y}(0) = \mathbf{Y}(0)$, $\mathbf{Z}(0) = \mathbf{Y}(0)$, so $\mathbf{Y}(0) = \mathbf{Z}(0)$.
- At $x = T$: $\mathbf{Y}(T) = \mathbf{Y}(0)$ (since $\mathbf{Y}(0) = \mathbf{Y}(T)$), and $\mathbf{Z}(T) = \mathbf{Y}(0)$, so $\mathbf{Y}(T) = \mathbf{Z}(T)$.

Thus, $\mathbf{W}$ satisfies the extended system (2.20) and the boundary conditions $\mathbf{Y}(0) = \mathbf{Z}(0)$, $\mathbf{Y}(T) = \mathbf{Z}(T)$.

Since both directions hold, the lemma is proved: $\mathbf{W}$ solves the extended system with the given boundary conditions if and only if $\mathbf{Y}$ satisfies $\mathbf{Y}' = \mathbf{A}(x, \lambda, \mu)\mathbf{Y}$ with $\mathbf{Y}(0) = \mathbf{Y}(T)$. $\square$

This extended system allows the definition of a periodic Evans function by analyzing solutions in the larger space, accounting for the non-separated boundary conditions introduced by periodicity.

Let $\boldsymbol{e_1} = (1, 0, 0, \ldots 0)$, $\boldsymbol{e_2} = (0, 1, 0, \ldots 0), \ldots, \boldsymbol{e_n} = (0, 0, 0, \ldots 1)$ be a canonical basis for $\mathbb{C}^n$, We set $\boldsymbol{b}_l^{\pm} = (\boldsymbol{e_l}, \boldsymbol{e_l})^\intercal$ for $l = 1, \ldots n$. and then let us introduce the boundary subspaces for the larger system:

$$\boldsymbol{W}(0) \in \mathbb{B}_0 := \operatorname{span}\{\boldsymbol{b_1^-}, \ldots \boldsymbol{b_n^-}\} \text{ and } \boldsymbol{W}(T) \in \mathbb{B}_T := \operatorname{span}\{\boldsymbol{b_1^+}, \ldots \boldsymbol{b_n^+}\}$$

We now have a dynamical system with separated boundary conditions. This would allow us to apply the already established theory for classical Evans function using Jost solutions.

It is easy to see that the fundamental solution matrix $\boldsymbol{\Phi}(x, \lambda, \mu)$ for ((2.11)) satisfies and induces a FMS for (2.20).

$$\boldsymbol{\Psi}(x, \lambda, \mu) = \begin{pmatrix} \boldsymbol{\Phi}(x, \lambda, \mu) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix}$$

This observation is necessary since the construction of Jost eigenfunctions $\boldsymbol{J}^{\pm}$ depends heavily on our understanding of the fundamental solution of the system.

On one side we have: $\boldsymbol{J}_j^-(x, \lambda, \mu) = \boldsymbol{\Psi}(x, \lambda, \mu)\boldsymbol{\Psi}^{-1}(0, \lambda, \mu)\boldsymbol{b}_j^- = \begin{pmatrix} \boldsymbol{\Phi}(x, \lambda, \mu)\boldsymbol{\Phi}^{-1}(0, \lambda, \mu)\boldsymbol{e_j} \\ \boldsymbol{e_j} \end{pmatrix}$

And on the other side: $\boldsymbol{J}_j^+(x, \lambda, \mu) = \boldsymbol{\Psi}(x, \lambda, \mu)\boldsymbol{\Psi}^{-1}(T, \lambda, \mu)\boldsymbol{b}_j^+ = \begin{pmatrix} \boldsymbol{\Phi}(x, \lambda, \mu)\boldsymbol{\Phi}^{-1}(T, \lambda, \mu)\boldsymbol{e_j} \\ \boldsymbol{e_j} \end{pmatrix}$

Definition 2.3.1 (The Periodic Evans Function)**.** We define the perodic Evans function by:

$$E(\lambda, \mu) = \det \begin{bmatrix} | & | & \cdots & | & | & \cdots & | \\ \boldsymbol{J}_1^- & \boldsymbol{J}_2^- & \cdots & \boldsymbol{J}_n^- & \boldsymbol{J}_1^+ & \cdots & \boldsymbol{J}_n^+ \\ | & | & \cdots & | & | & \cdots & | \end{bmatrix} (\frac{T}{2}, \lambda, \mu) \tag{2.22}$$

Lemma 2.3.2. The Evans function as defined in (2.22) is an entire function of λ for fixed μ and of μ for fixed λ. It has at most a countable number of zeros, with a zero at (λ, μ) if and only if λ is an eigenvalue of $\mathcal{L}_\mu$ defined in (2.15). Moreover, the multiplicity of roots of the Evans function is equal to the algebraic multiplicity of $\mathcal{L}_\mu$ at λ

Proof. The original system is $\mathbf{Y}' = \mathbf{A}(x, \lambda, \mu)\mathbf{Y}$, where $\mathbf{A}(x, \lambda, \mu) = \mathbf{A}(x, \lambda) - i\mu\mathbf{I}_n$ is an $n \times n$ matrix with smooth, T-periodic entries in x and analytic (polynomial) dependence on λ and μ. The extended system, designed to enforce periodic boundary conditions, is:

$$\mathbf{W}' = \begin{bmatrix} \mathbf{A}(x, \lambda, \mu) & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{bmatrix} \mathbf{W}, \quad \mathbf{W} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} \in \mathbb{C}^{2n},$$

with boundary conditions $\mathbf{Y}(0) = \mathbf{Z}(0)$, $\mathbf{Y}(T) = \mathbf{Z}(T)$. Denote the coefficient matrix as $\mathcal{A}(x, \lambda, \mu) = (\mathbf{A}(x, \lambda, \mu), \mathbf{0}_n)$, which is smooth in x and entire in λ, μ. The evolution operator $U(x, y; \lambda, \mu)$ for the extended system satisfies:

$$\frac{\partial}{\partial x} U(x, y; \lambda, \mu) = \mathcal{A}(x, \lambda, \mu)U(x, y; \lambda, \mu), \quad U(y, y; \lambda, \mu) = \mathbf{I}_{2n}.$$

Since $\mathcal{A}$ is smooth in x and entire in λ, μ, standard ODE theory (e.g., Picard-Lindelöf iteration) ensures that $U(x, y; \lambda, \mu)$ is smooth in x, y and entire in λ, μ (See Chapter 5 in [9] (Dependence on Initial Conditions and Parameters) and Chapter 2, Section 6 in [3] (Dependence on Initial Conditions and Parameters)) . Explicitly, $U(x, y; \lambda, \mu) = (\Phi(x, y; \lambda, \mu), \mathbf{I}_n)$, where $\Phi(x, y; \lambda, \mu)$ is the evolution operator for $\mathbf{A}(x, \lambda, \mu)$:

$$\frac{\partial}{\partial x} \Phi(x, y; \lambda, \mu) = \mathbf{A}(x, \lambda, \mu)\Phi(x, y; \lambda, \mu), \quad \Phi(y, y; \lambda, \mu) = \mathbf{I}_n.$$

Motivation: The evolution operator's analyticity is crucial for establishing that $E(\lambda, \mu)$ is entire, as it determines the behavior of solutions used in the Evans function.

The boundary conditions define two n-dimensional subspaces in $\mathbb{C}^{2n}$:

- At $x = 0$, $S^- = \{(\mathbf{Y}_0, \mathbf{Z}_0) \in \mathbb{C}^{2n} \mid \mathbf{Y}_0 = \mathbf{Z}_0\}$, with basis $\mathbf{v}_i^- = (\mathbf{e}_i, \mathbf{e}_i)$, $i = 1, \dots, n$, where $\mathbf{e}_i$ is the standard basis in $\mathbb{C}^n$.

- At $x = T$, $S^+ = \{(\mathbf{Y}_T, \mathbf{Z}_T) \in \mathbb{C}^{2n} \mid \mathbf{Y}_T = \mathbf{Z}_T\}$, with basis $\mathbf{v}_i^+ = (\mathbf{e}_i, \mathbf{e}_i)$, $i = 1, \dots, n$.

Propagate S^- forward to $x = T/2$:

$$U(T/2, 0; \lambda, \mu)S^- = \operatorname{span}\{\mathbf{J}_i^- = U(T/2, 0; \lambda, \mu)\mathbf{v}_i^- \mid i = 1, \dots, n\},$$

where:

$$\mathbf{J}_i^- = U(T/2, 0; \lambda, \mu) \begin{pmatrix} \mathbf{e}_i \\ \mathbf{e}_i \end{pmatrix} = \begin{pmatrix} \Phi(T/2, 0; \lambda, \mu)\mathbf{e}_i \\ \mathbf{e}_i \end{pmatrix}.$$

Propagate S^+ backward to $x = T/2$:

$$U(T/2, T; \lambda, \mu)S^+ = \operatorname{span}\{\mathbf{J}_i^+ = U(T/2, T; \lambda, \mu)\mathbf{v}_i^+ \mid i = 1, \dots, n\},$$

where:

$$\mathbf{J}_i^+ = U(T/2, T; \lambda, \mu) \begin{pmatrix} \mathbf{e}_i \\ \mathbf{e}_i \end{pmatrix} = \begin{pmatrix} \Phi(T/2, T; \lambda, \mu)\mathbf{e}_i \\ \mathbf{e}_i \end{pmatrix}.$$

Since $\Phi(x, y; \lambda, \mu)$ is entire in λ, μ, the vectors $\mathbf{J}_i^\pm$ are entire in λ, μ. The periodic Evans function is:

$$E(\lambda, \mu) = \det \begin{bmatrix} \mathbf{J}_1^- & \cdots & \mathbf{J}_n^- & \mathbf{J}_1^+ & \cdots & \mathbf{J}_n^+ \end{bmatrix} (T/2, \lambda, \mu).$$

Here, we note that the subspaces S^- and S^+ encode the periodic boundary conditions via the extended system. The Evans function measures their intersection at $x = T/2$, detecting non-trivial solutions satisfying $\mathbf{Y}(0) = \mathbf{Y}(T)$.

To show analyticity, we begin by observing that the matrix

$$\begin{bmatrix} \mathbf{J}_1^- & \cdots & \mathbf{J}_n^- & \mathbf{J}_1^+ & \cdots & \mathbf{J}_n^+ \end{bmatrix}$$

has entries that are entire in λ for fixed μ and in μ for fixed λ, since $\Phi(x, y; \lambda, \mu)$ is entire and the basis vectors $\mathbf{e}_i$ are constant. The determinant $E(\lambda, \mu)$ is a polynomial in these entries, hence entire in λ for fixed μ and in μ for fixed λ.

Recall that analyticity is inherited from the ODE's parameter dependence, ensuring $E(\lambda, \mu)$ is well-behaved for complex analysis.

Countable Number of Zeros: An entire function in λ for fixed μ (or vice versa) that is not identically zero has at most countably many zeros, which are isolated unless the function is zero everywhere (by the identity theorem for analytic functions). The zeros may accumulate only at infinity in the complex plane. For a fixed μ, the operator under consideration possesses a compact resolvent, which implies that its spectrum is discrete. This result is analogous to the general proof of the spectral theorem for second-order ODEs, as detailed in [10]

The Evans function $E(\lambda, \mu) = 0$ if and only if the columns $\mathbf{J}_1^-, \dots, \mathbf{J}_n^-, \mathbf{J}_1^+, \dots, \mathbf{J}_n^+$ are linearly dependent at $x = T/2$, i.e.,

$$\dim(U(T/2, 0; \lambda, \mu)S^- \cap U(T/2, T; \lambda, \mu)S^+) > 0.$$

This occurs if there exists a non-trivial $\mathbf{W} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} \in S^-$ at $x = 0$ such that $U(T, 0; \lambda, \mu)\mathbf{W} \in S^+$ at $x = T$.

By Lemma (2.3.1), this is equivalent to $\mathbf{Y}$ satisfying $\mathbf{Y}' = \mathbf{A}(x, \lambda, \mu)\mathbf{Y}$ with $\mathbf{Y}(0) = \mathbf{Y}(T)$. Since $\mathbf{A}(x, \lambda, \mu)$ is the companion matrix for $\mathcal{L}_\mu q = \lambda q$, where $\mathcal{L}_\mu$ acts on T-periodic functions, a non-trivial $\mathbf{Y}$ with $\mathbf{Y}(0) = \mathbf{Y}(T)$ corresponds to a non-trivial q such that $\mathcal{L}_\mu q = \lambda q$. Thus,

$E(\lambda, \mu) = 0$ if and only if λ is an eigenvalue of $\mathcal{L}_\mu$. Again, we recall that the Evans function detects non-trivial intersections of boundary subspaces, which, via the extended system, correspond to solutions of the periodic BVP for $\mathcal{L}_\mu$.

Claim: Multiplicity Equals Algebraic Multiplicity. The algebraic multiplicity of an eigenvalue λ_0 of $\mathcal{L}_\mu$ on $H^n([0, T])$ with periodic boundary conditions is the dimension of the generalized kernel $\ker(\mathcal{L}_\mu - \lambda_0 I)^k$ for large k. By results in Evans function theory for periodic problems (e.g., [10, pp. 68–70], [7]), the order of the zero of $E(\lambda, \mu)$ at λ_0 equals this algebraic multiplicity. □

Lemma 2.3.3 (Alternative Definition)**.** Let $\Phi(x, \lambda, \mu)$ be a fundamental matrix solution to the periodic system. The Evans function defined by

$$E(\lambda, \mu) = \det\left(\boldsymbol{\Phi}(T, \lambda, \mu)\boldsymbol{\Phi}^{-1}(0, \lambda, \mu) - \boldsymbol{I}_n\right) \tag{2.23}$$

shares the same properties stated in the previous lemma for the definition (2.22)

Remark 2.3.1. The first definition (2.22) is constructed by a determinant that vanishes once periodic eigenfunctions exist. Similarly, the definition (2.23) approaches the problem behind a lens of Floquet theory. The alternative method focuses on the eigenvalues of the monodromy matrix $\boldsymbol{\Phi(T)\Phi^{-1}}(0)$ which describes how the solutions evolve over one period. The determinant defined in the alternative approach checks if the monodromy matrix has 1 as eigenvalue, implying that a solution repeats exactly after period T, leading to a periodic eigenfunction.

Remark 2.3.2 (Analyticity of the alternative case)**.** Recall that the monodromy matrix itself is a product of fundamental solutions, which are analytic, hence the alternative definition also produces an entire function.

2.3.3 Computations with Periodic Evans Function

A topological approach is often considered when directly locating eigenvalues proves challenging. In particular, we employ a winding number as a tool to locate eigenvalues of the operator and analyze spectral properties. By the argument principle, the winding number of the periodic Evans function $E(\lambda, \mu)$ around a contour counts the number of its zeros, corresponding to eigenvalues of $\mathcal{L}_\mu$ defined in (2.15), with their algebraic multiplicities.

Theorem 2.3.1. Consider the operator $\mathcal{L}$ as defined in (2.10). Let $C \subset \mathbb{C}$ be a positively oriented simple closed curve. The curve is assumed not to intersect $\sigma(\mathcal{L})$. The winding number

$$W(\mu) = \frac{1}{2\pi i} \oint_C \frac{\partial_\lambda E(\lambda, \mu)}{E(\lambda, \mu)} d\lambda$$

is constant for $\mu \in (-1, 1]$. Moreover, if $W(0) = 1$, then the spectra inside of C forms a smooth, closed curve.

Remark 2.3.3 (Intuition and Further Remarks on the Winding Number)**.** We are essentially investigating how the eigenvalues of the operator $\mathcal{L}$ change with respect to the parameter μ. While observing the problem from a dynamical systems perspective, we develop a topological tool using a contour integral. Since zeros of the periodic Evans function $E(\lambda, \mu)$ correspond to eigenvalues of the linearized operator, we consider the encirclement of zero in the complex plane to compute how many eigenvalues are enclosed by the curve C. The winding number turns out to be constant for specific values of μ. That is, if $\mu \in (-1, 1]$, then the constant winding number suggests that the enclosed eigenvalues are entirely within C and follow a continuous, smooth path as μ varies. This is particularly useful, since it means that the information is predictable, and hence no eigenvalues suddenly appear or disappear in this range.

Remark 2.3.4 (Implications of $W(0) = 1$)**.** Spectral curves appear once the initial winding number $W(0) = 1$, which implies that the distribution of the eigenvalues will not manifest itself as chaotic, but it will rather evolve predictably with μ leading to a closed spectral curve.

A useful extension to the already-established results is presented by the following theorem, which claims that once we expand the periodicity of the problem, the behavior in the structure of eigenvalues still remains predictable.

Theorem 2.3.2. *Consider the original eigenvalue problem (2.10) together with periodic boundary conditions on $[0, \pi]$. As before, let the curve $C \subset \mathbb{C}$ be positively oriented and simple closed that does not intersect the spectrum $\sigma(\mathcal{L})$. If $W(0) = m$ for π-periodic Bloch-wave problem, then for each $\frac{-1}{k} < \mu < \frac{1}{k}$ the corresponding $k\pi$-periodic Bloch-wave problem satisfies $W(\mu) = km$.*

Proof. For the π-periodic problem ($\mu = 0$), the Evans function is:

$$E(\lambda, 0) = \det(\Phi(\pi, \lambda, 0) - I),$$

where $\Phi(\pi, \lambda, 0)$ is the monodromy matrix as defined earlier. The winding number $W(0) = m$ implies m eigenvalues inside C.

For the $k\pi$-periodic problem, we seek solutions satisfying:

$$\mathbf{y}(x + k\pi) = e^{i\mu k\pi}\mathbf{y}(x),$$

with $\mu \in \left(-\frac{1}{k}, \frac{1}{k}\right)$. Since $\mathbf{y}(x + \pi) = \Phi(\pi, \lambda, 0)\mathbf{y}(x)$, after k periods:

$$\mathbf{y}(x + k\pi) = \Phi(\pi, \lambda, 0)^k\mathbf{y}(x).$$

The Bloch condition requires:

$$\det(\Phi(\pi, \lambda, 0)^k - e^{i\mu k\pi}I) = 0.$$

Thus, the Evans function for the $k\pi$-periodic problem is:

$$E_k(\lambda, \mu) = \det(\Phi(\pi, \lambda, 0)^k - e^{i\mu k\pi}I).$$

Let the eigenvalues of $\Phi(\pi, \lambda, 0)$ be $\rho_1(\lambda), \ldots, \rho_n(\lambda)$. Then $\Phi(\pi, \lambda, 0)^k$ has eigenvalues $\rho_j(\lambda)^k$. The equation $D_k(\lambda, \mu) = 0$ holds when:

$$\rho_j(\lambda)^k = e^{i\mu k\pi},$$

i.e.:

$$\rho_j(\lambda) = e^{i\mu\pi}e^{i2\pi\ell/k}, \quad \ell = 0, 1, \ldots, k - 1.$$

For each eigenvalue λ_j of the π-periodic problem, where $\rho_j(\lambda_j) = 1$, there are k solutions to $\rho_j(\lambda)^k = e^{i\mu k\pi}$, corresponding to the k-th roots of $e^{i\mu k\pi}$. Since $W(0) = m$ implies m eigenvalues inside C, each generates k eigenvalues for the $k\pi$-periodic problem.

The winding number $W(\mu)$ counts the zeros of $E_k(\lambda, \mu)$ inside C. Since each of the m eigenvalues splits into k eigenvalues, and C does not intersect $\sigma(\mathcal{L})$, the total number of zeros is:

$$W(\mu) = k \cdot m.$$

∎

Remark 2.3.5. The above theorem essentially extends the results of the preceding discussion on the winding number to the Bloch-wave spectral problem. The interesting result is the fact that the winding number scales itself accordingly as the periodicity of the problem evolves to being $k\pi$-periodic from its original π-periodic form. Having $W(0) = m$ implies that the Evans function $E(\lambda, 0)$ encircles the origin exactly m times along the contour C. As an intuitive suggestion, we are essentially stating that the number of eigenvalues (including algebraic multiplicities) enclosed by the contour is indeed proportional to m.

Remark 2.3.6 (Increase of the periodic domain)**.** Transitioning to a $k\pi$ periodic problem in Bloch-wave form increases the periodic domain length by a factor of k. Particularly of interest is the fact that the eigenvalues of $\mathcal{L}$ are thereby rescaled accordingly. The claim that $W(\mu) = kW(0)$ implies that the spectral structure of the $k\pi$ periodic problem is akin to having exactly k copies of the original spectrum, each repeating the same topological winding.

Remark 2.3.7. Since the winding number tracks the number of eigenvalues enclosed by the contour C, we arrive at the following conclusion:

Increasing periodicity $\implies$ increase in eigenvalues

In principle, we have developed a scaling law for spectral topology in Bloch-wave problems.

2.4 Applications To The Nonlinear Schrödinger Equation

2.5 Historical Background

Quantum mechanics is a fundamental theory in physics that describes the behavior of matter and energy at very small scales, such as atoms and subatomic particles. Unlike classical physics, where objects follow predictable paths, quantum mechanics uses a wave function $\psi(x,t)$ to represent the state of a particle. The wave function is a complex-valued function whose magnitude squared, $|\psi(x,t)|^2$, gives the probability of finding the particle at position x at time t. The evolution of $\psi(x,t)$ is governed by the Schrödinger equation, which for a single particle in one dimension is:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi, \tag{2.24}$$

where $\hbar$ is the reduced Planck constant, m is the particle's mass, $V(x)$ is the potential energy (e.g., due to an external force field). The linearity of this equation results in superposed solutions, and the equation itself describes non-interacting particles in small systems such as atoms or quantum wells.

The nonlinear Schrödinger (NLS) equation arises when interactions between particles or between a wave and its medium are significant, introducing nonlinearity into the wave dynamics. In this chapter, we consider the Nonlinear Schrödinger Equation (NLS) in one spatial dimension, given by

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad u(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}, \quad t > 0. \tag{2.25}$$

and analyze the spectral stability of plane wave solutions.

2.6 Application of the Periodic Evans Function to Stability Analysis of Plane Wave Solutions in the NLS Equation

Our goal is to compute the Jost solutions, construct the Evans matrix, and apply the periodic Evans function framework developed earlier to assess the spectral stability of periodic traveling wave solutions to the Nonlinear Schrödinger Equation (NLS). We focus exclusively on plane wave solutions, excluding solitons, and consider both focusing and defocusing cases, drawing on results from [6, 5, 8, 1].

2.6.1 NLS and Plane Wave Solutions

The Nonlinear Schrödinger Equation (NLS) is given by:

$$iu_t + u_{xx} \pm |u|^2 u = 0, \tag{2.26}$$

where the sign is + for the focusing case and − for the defocusing case. Plane wave solutions are of the form:

$$u(x,t) = ae^{i(kx-\omega t)},$$

with amplitude $a > 0$, wave number $k \in \mathbb{R}$, and frequency $\omega = k^2 \pm a^2$. These solutions are periodic in x with arbitrary period $T > 0$, as $e^{ikx} = e^{ik(x+T)}$ when $kT = 2\pi m$ for some integer m. For simplicity, we choose a frame where $k = 0$, so:

$$u_0(x,t) = ae^{\pm ia^2 t},$$

which is constant in x, hence periodic with any period T. We fix $T = 2\pi$ for concreteness, aligning with standard Floquet analysis.

2.6.2 Linearization and Bloch-Wave Decomposition

To assess spectral stability, we linearize the NLS around u_0. Consider a perturbation $u(x,t) = u_0(x,t)(1 + v(x,t))$, where v is small. Substituting into (2.26) and retaining linear terms, the linearized equation is:

$$iv_t + v_{xx} \pm a^2(v + \overline{v}) = 0. \tag{2.27}$$

Separating real and imaginary parts, we write $v = v_r + iv_i$, where $v_r, v_i \in \mathbb{R}$. Plugging this into (2.27), we'll have:

$$i\left(v_r + iv_i\right)_t + v_{rxx} + iv_{ixx} \pm 2a^2 v_r = 0$$

$$v_{rt} + v_{ixx} = 0$$

$$-v_{it} + v_{rxx} \pm 2a^2 v_r = 0$$

$$\begin{pmatrix} v_r \\ v_i \end{pmatrix}_t = \begin{pmatrix} 0 & -\partial_{xx} \\ \partial_{xx} \pm 2a^2 & 0 \end{pmatrix} \begin{pmatrix} v_r \\ v_i \end{pmatrix}$$

The operator $\mathcal{L}$ is defined as:

$$\mathcal{L} = \begin{pmatrix} 0 & -\partial_{xx} \\ \partial_{xx} \pm 2a^2 & 0 \end{pmatrix},$$

acting on $H^2([0,T]) \times H^2([0,T])$ with periodic boundary conditions. Spectral stability requires that the spectrum $\sigma(\mathcal{L})$ lies on the imaginary axis ($\mathrm{Re}(\lambda) = 0$); otherwise, eigenvalues with $\mathrm{Re}(\lambda) > 0$ indicate instability.

Since $\mathcal{L}$ has constant coefficients (due to the plane wave's spatial uniformity), we use Bloch-wave decomposition. Assume a solution of the form:

$$\begin{pmatrix} v_r(x,t) \\ v_i(x,t) \end{pmatrix} = e^{i\mu x + \lambda t} \begin{pmatrix} q_r(x) \\ q_i(x) \end{pmatrix},$$

where q_r, q_i are T-periodic, and $\mu \in [-1, 1]$ is the Bloch parameter (normalized for $T = 2\pi$). The eigenvalue problem $\mathcal{L}\mathbf{q} = \lambda\mathbf{q}$ becomes:

$$\lambda \begin{pmatrix} q_r \\ q_i \end{pmatrix} = \begin{pmatrix} 0 & -(\partial_x + i\mu)^2 \\ (\partial_x + i\mu)^2 \pm 2a^2 & 0 \end{pmatrix} \begin{pmatrix} q_r \\ q_i \end{pmatrix} = \mathcal{L}_\mu \mathbf{q}.$$

2.6.3 Jost Solutions and Evans Function

We begin by converting $\mathcal{L}_\mu \mathbf{q} = \lambda \mathbf{q}$ to a first-order system. Let $\mathbf{Y} = (q_r, q_i, \tilde{q}_r, \tilde{q}_i)^T$, such that:

$$\begin{aligned}
\tilde{q}_r &= (\partial_x + i\mu)\, q_r \\
\tilde{q}_i &= (\partial_x + i\mu)\, q_i \\
(\partial_x + i\mu)\, \tilde{q}_r = (\partial_x + i\mu)^2\, q_r &= \lambda q_i \mp 2a^2 q_r \\
\partial_x \tilde{q}_r &= -i\mu \tilde{q}_r + \lambda q_i \mp 2a^2 q_r \\
(\partial_x + i\mu)\, \tilde{q}_i = (\partial_x + i\mu)^2\, q_i &= -\lambda q_r \\
\partial_x \tilde{q}_i &= -i\mu \tilde{q}_i - \lambda q_r
\end{aligned}$$

The dynamical system will therefore be of the form:

$$\mathbf{Y}' = \underbrace{\begin{pmatrix} -i\mu & 0 & 1 & 0 \\ 0 & -i\mu & 0 & 1 \\ \mp 2a^2 & \lambda & -i\mu & 0 \\ -\lambda & 0 & 0 & -i\mu \end{pmatrix}}_{\mathbf{A}(\lambda,\mu)} \mathbf{Y}.$$

Note that $\mathbf{A}(\lambda, \mu) = \mathbf{A}(\lambda, 0) - i\mu\mathbf{I}$. So the eigenvalues of $\mathbf{A}(\lambda, \mu)$ are the same as the eigenvalues of $\mathbf{A}(\lambda, 0)$, shifted by $i\mu$ The Jost solutions are the columns of the principal fundamental matrix solution $\Phi(x, \lambda, \mu) = e^{\mathbf{A}(\lambda,\mu)x}$, since $\mathbf{A}$ is constant. The monodromy matrix is:

$$\mathbf{M}(\lambda, \mu) = \Phi(T, \lambda, \mu) = e^{\mathbf{A}(\lambda,\mu)T}.$$

From the alternative definition, the Evans function is:

$$E_2(\lambda, \mu) = \det\left(\mathbf{M}(\lambda, \mu) - \mathbf{I}_4\right) = \det\left(e^{\mathbf{A}(\lambda,\mu)T} - \mathbf{I}_4\right),$$

which is related to $E_1(\lambda, \mu)$ from (2.22) by:

$$E_2(\lambda, \mu) = \det\left(\Phi(T, \lambda, \mu)\Phi^{-1}(T/2, \lambda, \mu)\right) E_1(\lambda, \mu).$$

Since $\det\left(\Phi(T)\Phi^{-1}(T/2)\right) = \det\left(e^{\mathbf{A}(\lambda,\mu)T/2}\right)$ is non-zero and analytic, E_2 and E_1 share the same zeros and multiplicities.

2.6.4 The Evans Function for NLS

We follow what we previously discussed regarding the computations of the Evans function, beginning by constructing the characteristic polynomial:

$$\det(\mathbf{A}(\lambda, \mu) - \nu\mathbf{I}_4) = \det(\mathbf{A}(\lambda, 0) - (\nu + i\mu)\,\mathbf{I}_4) = 0.$$

$$\nu + i\mu \text{ is an eigenvalue of } \mathbf{A}(\lambda, 0) \iff \nu \in \sigma(\mathbf{A}(\lambda, 0) - i\mu$$

The matrix $\mathbf{A}(\lambda, 0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mp 2a^2 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \end{bmatrix}$ has characteristic polynomial $\pm 2a^2\nu^2 + \nu^4 + \lambda^2 = 0$.

We write it in quadratic form and use it later

$$(\nu^2)^2 \pm 2a^2\nu^2 + \lambda^2 = 0 \tag{2.28}$$

Similarly for $\mathbf{A}(\lambda, \mu)$ we have the following characteristic polynomial:

$$2a^2(\nu + i\mu)^2 + (\nu + i\mu)^4 + \lambda^2 = 0$$

$$\iff \nu^2 = \mp a^2 \pm \sqrt{a^4 - \lambda^2} \text{ and } (\nu + i\mu)^2 = \mp a^2 \pm \sqrt{a^4 - \lambda^2}$$

Fixing $T = 2\pi,$ and $\nu = im, m \in \mathbb{Z}$, from (2.28) we have

$$(m + \mu)^4 \mp 2a^2(m + \mu)^2 + \lambda^2 = 0.$$

2.6.5 Focusing Case

$$\lambda^2 = \left(2a^2 - (m+\mu)^2\right)(m+\mu)^2$$

We will have $\Re(\lambda) > 0$ if $(m+\mu)^2 < 2a^2$, which would indicate instability of the inequality holds.

2.6.6 Defocusing Case

$$\lambda^2 = -\left(2a^2 + (m+\mu)^2\right)(m+\mu)^2 \leq 0$$

The defocusing case will never experience any instability, as the above inequality indicates negative real parts all the time.

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.1 Alternative Proof for Theorem (2.3.1)

Proof. To prove the theorem, we use the framework of Kapitula and Promislow [**?**, Theorem 8.4.3], generalizing the result to the case where the winding number $W(0) = m$ for the π-periodic Bloch-wave problem. We show that for the $k\pi$-periodic problem, the winding number is $W(\mu) = km$ for $\mu \in \left(-\frac{1}{k}, \frac{1}{k}\right)$.

The operator $\mathcal{L} : H^n([0,\pi]) \to L^2([0,\pi])$ with periodic boundary conditions is defined in (2.10). For the Bloch-wave problem, we consider solutions of the form

$$p(x) = e^{i\mu x} q(x),$$

where $q(x)$ is π-periodic, and $\mu \in [-1,1]$ is the Bloch parameter, normalized to the Brillouin zone for period π. The eigenvalue problem $\mathcal{L}p = \lambda p$ becomes

$$\mathcal{L}_\mu q = \left(\left(\frac{d}{dx} + i\mu\right)^n + a_{n-1}(x)\left(\frac{d}{dx} + i\mu\right)^{n-1} + \cdots + a_1(x)\left(\frac{d}{dx} + i\mu\right) + a_0(x)\right) q = \lambda q,$$

where $\mathcal{L}_\mu$ acts on π-periodic functions in $H^n([0,\pi])$. The periodic Evans function $E(\lambda,\mu)$ is defined so that its zeros correspond to eigenvalues of $\mathcal{L}_\mu$, with multiplicity. The winding number is

$$W(\mu) = \frac{1}{2\pi i} \oint_C \frac{\partial_\lambda E(\lambda,\mu)}{E(\lambda,\mu)}\, d\lambda,$$

which, by the argument principle, counts the number of eigenvalues of $\mathcal{L}_\mu$ inside C, including multiplicities. By assumption, $W(0) = m$.

From [10, Theorem 8.4.3], for the π-periodic Bloch-wave problem, the winding number $W(\mu)$ is constant in each interval $(l/k, (l+1)/k)$ for $l = 0, \ldots, k-1$, assuming the spectrum inside C consists of m branches forming smooth curves. Since $W(0) = m$, and assuming the spectrum is non-degenerate, $W(\mu) = m$ in each such interval for the π-periodic problem. This constancy arises because the eigenvalues of $\mathcal{L}_\mu$ vary analytically with μ, and C does not intersect $\sigma(\mathcal{L})$, ensuring no eigenvalues cross the contour.

For the $k\pi$-periodic problem, we consider solutions $p(x) = e^{i\mu x} q(x)$, where $q(x)$ is $k\pi$-periodic, and the operator $\mathcal{L}$ is now defined on $[0, k\pi]$ with periodic boundary conditions. The corresponding Bloch operator $\mathcal{L}_{k,\mu}$ acts on $k\pi$-periodic functions, and the Evans function $E_k(\lambda,\mu)$ detects its eigenvalues. The key insight from Kapitula and Promislow is that the $k\pi$-periodic problem can be analyzed by relating it to the π-periodic problem. Specifically, a $k\pi$-periodic function $q(x)$ can be decomposed into π-periodic components via Bloch waves with shifted parameters:

$$E_k(\lambda,\mu) = \prod_{l=0}^{k-1} E\left(\lambda, \mu + \frac{l}{k}\right),$$

up to a nonzero analytic factor that does not affect the winding number.

The winding number for the $k\pi$-periodic problem is

$$W_k(\mu) = \frac{1}{2\pi i} \oint_C \frac{\partial_\lambda E_k(\lambda,\mu)}{E_k(\lambda,\mu)}\, d\lambda.$$

Using the product form,

$$\partial_\lambda \log E_k(\lambda,\mu) = \sum_{l=0}^{k-1} \partial_\lambda \log E\left(\lambda, \mu + \frac{l}{k}\right),$$

we obtain

$$W_k(\mu) = \sum_{l=0}^{k-1} \frac{1}{2\pi i} \oint_C \frac{\partial_\lambda E(\lambda, \mu + l/k)}{E(\lambda, \mu + l/k)}\, d\lambda = \sum_{l=0}^{k-1} W\left(\mu + \frac{l}{k}\right).$$

For $\mu \in \left(-\frac{1}{k}, \frac{1}{k}\right)$,

$$\mu + \frac{l}{k} \in \left(\frac{l-1}{k}, \frac{l+1}{k}\right), \quad l = 0, \dots, k-1.$$

By the constancy result above, $W(\mu + l/k) = m$ in each such interval, hence

$$W_k(\mu) = \sum_{l=0}^{k-1} m = km.$$

The zeros of $E_k(\lambda, \mu)$ correspond to eigenvalues of $\mathcal{L}_{k,\mu}$, and each zero of $E(\lambda, \mu + l/k)$ indicates an eigenvalue of $\mathcal{L}_{\mu+l/k}$. The multiplicity sum confirms that $W_k(\mu) = km$ counts the total number of eigenvalues inside C. $\square$

Bachelor’s Theses in Mathematical Sciences 2025:K24
ISSN 1654-6229

LUNFMA-4178-2025

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